PROBLEMS

IN

HIGH SPEED FLOW

Thesis

submitted by

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PREFACE

I am deeply indebted to Professor D. C. Pack for his introduction to this research topic and wish to record my thanks both for his continued interest in the problem and for his supervision and guidance in regard to difficulties encountered.

I wish also to thank Dr. R. Schlapp and Mr. A. Nisbet for their advice, discussion and encouragement.

The content of this thesis is declared to be original except where reference is made to other work.

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INTRODUCTION

This thesis is concerned with the problem of compound gas jets, that is a gas jet embedded in a gaseous stream of finite width. In all cases the gas is presumed to be compressible, but the stream and jet may be either subsonic or supersonic.

To make the problem tractable it is assumed that the pressure difference between jet and stream is small so that, as a consequence, the fluctuations of the jet boundary are small and the method of small perturbations may be used to linearise both the steady equations of motion and the boundary conditions of the problem. Heat conduction and fluid viscosity are neglected.

The investigation of a subsonic jet of gas issuing from an orifice into an infinite medium at rest has been undertaken by several writers (1)(2)(3)(4). This type of flow is characterised by a monotonic structure in the jet and the contraction in the cross section of the jet (vena contracta). Much attention has also been given to the problem of a two-dimensional supersonic gas jet issuing into an infinite medium at rest (5)(6)(7)(8)(9)(10), and to an axially symmetrical jet issuing into an infinite region at rest (5)(11)(12). This type of flow is characterised, for small excess pressure in the jet, by a periodic (or almost periodic) structure in the jet and the appearance of expansive and compressive waves.

The problem of a gas jet issuing into an infinite uniform stream has, however, only recently been studied. That of a supersonic jet which emerges into a subsonic stream has been considered by Pai (13) and that of a supersonic jet in a supersonic stream by

Kawamura [14], Pack [15] and Ehlers and Strand [16]. The boundary conditions of the problem discussed by Pai were not fully specified, no account being taken of the fact that the disturbances in the subsonic stream would spread upstream as well as down. This omission in the boundary conditions meant that the solutions obtained could only be valid far downstream of the jet exit. This fact was noted by Klunker and Harder [17] who, without further comment, quoted a solution in integral equation form but made no attempt to present the solution in an exact form.

While the present author was investigating this problem it was suggested by Professor Pack that the solution might be obtained by employing an analysis based on the Wiener-Hopf technique. On carrying out the suggestion it was found that the technique failed due to the appearance of a non-analytic function (of the form $\frac{\alpha}{|\alpha|}$, where α is complex) in the analysis. This situation arises whenever the Wiener-Hopf technique is applied to problems in which the variable satisfies a Laplace type field equation (i.e. the velocity potential in the subsonic region in this problem). It was found that there were two ways in which this difficulty could be avoided. One way was to introduce some damping into the flow in the subsonic region. The effect of the damping term was to replace the non-analytic term by an analytic one (of the form $\frac{\alpha}{\gamma}$, where $\gamma = \sqrt{(\alpha^2 - k^2)}$ and k is real). The problem was then capable of solution by the Wiener-Hopf technique. From the general solution, the damping effect would be removed by a limiting process and the solution to the original problem obtained. The second way of avoiding the non-analytic term was to introduce rigid walls and restrict the width of the outer subsonic stream. It then appeared that all of the functions arising were analytic and the problem was solved. The solution to the original problem was obtained by letting the walls recede steadily to an infinite distance. In this thesis the second method of approach to the problem is adopted partly because of the simpler Wiener-Hopf type equation which arises and partly because it was felt that the compound jet had more practical significance than a flow with an artificial damping effect.

After dealing with this problem of a supersonic jet embedded in a subsonic flow it was found that the technique employed could be easily modified to deal with other problems of compound gas jets both in the two-dimensional and the axially symmetrical cases.

The first part of the thesis, consisting of Chapters I to IV, deals with the two-dimensional problems and the second part, consisting of Chapters V to VIII, with the axially symmetrical problems. It is shown that in all cases it is possible to produce solutions, in infinite series form, for the various regions of flow, subject always to the restrictions and approximations of the linearised theory.

Chapter I is concerned with the solution of the original problem. It is shown there that the solution obtained by Pai is just the asymptotic form of the general solution. The initial behaviour of the jet boundary and the ultimate jet width are examined. It is found that the nature of the solution is different according as a certain relationship between the gas constants, Mach numbers and widths of the streams is greater than, equal to or less than unity. In particular for the case when this relationship is unity it is shown that the solution leads to large vaues of the perturbation velocities downstream of the jet exit. This solution is inconsistent with the assumption of a linearised theory. The case is compared in some detail with another problem, discussed by Stoker[25], dealing with water waves on the surface of a running stream. From the discussion it is concluded that there is a breakdown in the assumption of a linearised theory for the critical value of the above relationship.

Chapter II deals with a supersonic jet embedded in a supersonic stream and the results extend those already obtained by Pack. In his problem Pack shows that there is a certain critical value of a second relationship between the gas constants and the Mach numbers of the streams for which the disturbances within the jet are all transmitted through the jet boundary and none reflected. The boundary of the jet expands to a certain width and then remains constant thereafter. In this chapter, this case is examined for various stream widths and it is shown exactly how the transition is made from the flow with an outer stream of finite width to that of an outer stream of infinite width. Also in this chapter, the general solution is obtained by the method of characteristics and the reflection and transmission coefficients found. The text is illustrated with some diagrams.

The first part of the thesis concludes with Chapters III and IV which deal with the problems of a subsonic jet embedded in a supersonic stream and in a subsonic stream, respectively. A discussion into the behaviour of the jet boundary far downstream, in each of the four problems, is given at the end of Chapter IV where it is shown that the results obtained are in agreement with those which can be predicted on physical grounds.

The solutions for the axially symmetrical compound jet problems of Chapter V to VIII are found to be more complicated than those of the corresponding two-dimensional solutions of Chapters I to IV respectively but, nevertheless, the basic mathematical procedures are shown to be identical. This follows from the similarity between the trigonometric and hyperbolic functions appearing in the two-dimensional problems and the combinations of Bessel functions appearing in the axially symmetrical problems. The form of the solutions to the problems of Part II is similar to that of the solutions of the corresponding problems of Part I but it is shown that there are two fundamental differences. The first is in the magnitudes of the displacements of the jet boundary and it is shown, in the case of an infinite outer stream, that this displacement in the axially symmetrical problems is one half of that of the corresponding two-dimensional problems. The second difference is in the appearance of discontinuities in the velocity across the Mach cones in the supersonic flow. When the jet is supersonic these discontinuities culminate in an infinite singularity at the vertex of the Mach cone. Some remarks are made in the conclusions to Chapters V and VI concerning these singularities and the focussing effect of axially symmetrical flows.

At the end of the dissertation are collected, in appendix form, results required for use in the various problems.

CHAPTER I

The flow of a supersonic two-dimensional gas jet in a uniform subsonic flow

1. Formulation of the problem and the general solution

The original investigation of the problem of a supersonic jet flowing into an infinite subsonic stream was undertaken by Pai [13] who came to the conclusion that the flow pattern in the supersonic jet was almost periodic. In the formulation of the problem the boundary conditions were not fully specified, no account being taken of the fact that the disturbance in the subsonic stream would spread upstream as well as downstream. The solution obtained by Pai was thus only valid far downstream of the jet exit. This fact was noted by Klunker and Harder [17] who, in a comment on Pai's paper, indicated the correct boundary conditions and derived the solution of the problem in an integral equation form.

In this chapter the problem of a two-dimensional jet issuing from an orifice into a finite subsonic stream will be reconsidered and a solution in an exact form obtained by re-formulating the field equations and boundary conditions and making use of the Wiener-Hopf technique [18][19]. The solution obtained by Pai will be shown to be the asymptotic form of this solution.

Suppose an ideal gas to pass through a straight walled nozzle $-h < y < +h$; $-\infty < x < 0$ and issue as a two-dimensional supersonic jet, in the region $-h < y < +h$, $0 < x < +\infty$, into a subsonic stream of ideal gas, in the region $h < |y| < H$; $-\infty < x < +\infty$, which flows in the same direction as the jet. The stream is bounded by rigid walls at $y = \pm H$. Figure 1 shows a schematic diagram of the flow regions. The line $y = 0$ is a line of symmetry and hence it is only necessary to consider the region $y \geqslant 0$.

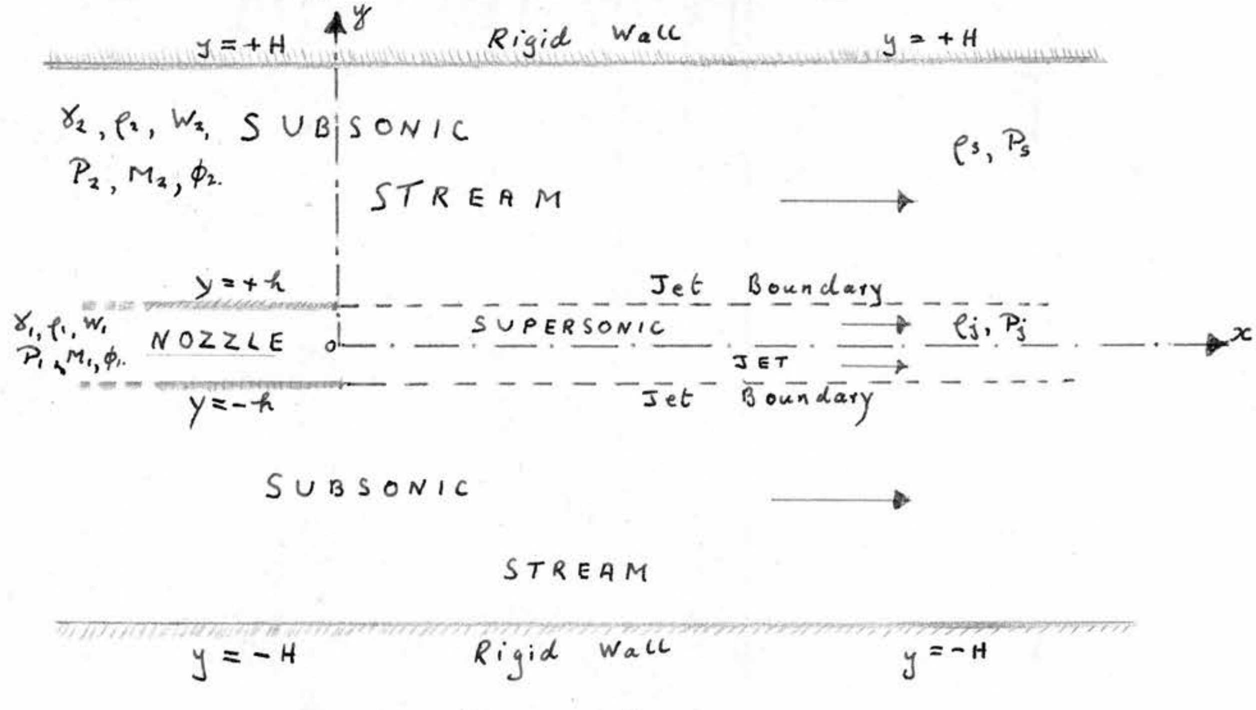

Figure. 1.

Let the suffix 1 refer to the jet and the suffix 2 to the outer stream. The jet is a uniform parallel stream of gas, of specific heat ratio γ_1, which has a velocity, in the nozzle, of W_1 with Mach number $M_1 (>1)$ and density ρ_1 with pressure P_1. Similarly the outer stream consists of a gas, with ratio of specific heats γ_2, which has a velocity, in the undisturbed state, of W_2 and density ρ_2 with pressure P_2. In both media heat conduction and viscosity are neglected and a steady state assumed.

The pressure difference $P_1 - P_2$ is small compared with P_1, P_2 or $\rho_2 W_2^2$ and the variation of the two streams from their original parallel flow is small so that the linearised theory may be used.

The velocity potential in the jet and outer stream respectively, may be written

$$\Phi_1(x,y) = W_1\left[x + \phi_1(x,y)\right] \quad \text{and} \quad \Phi_2(x,y) = W_2\left[x + \phi_2(x,y)\right] \tag{1}$$

The functions $\phi_1(x,y)$ and $\phi_2(x,y)$ will be called the perturbation potentials.

The boundary conditions are formed by using the following conditions: that there is no flow over the jet boundary, that there is no flow over either the rigid walls or the axis of symmetry and that there is continuity in the pressure across the jet boundary.

The condition for continuity in the direction of flow across the jet boundary is given by

$$\frac{\partial \phi_2}{\partial y} = \frac{\partial \phi_1}{\partial y} \quad \text{on} \quad y = h, \quad 0 < x < \infty, \tag{2}$$

using the approximations of the linearised theory. The condition that there is no flow over the rigid walls is given by

$$\frac{\partial \phi_2}{\partial y} = \frac{\partial \phi_1}{\partial y} = 0, \quad \text{on } y = h, \quad -\infty < x < 0 \tag{3}$$

and
$$\frac{\partial \phi_2}{\partial y} = 0, \quad \text{on } y = H, \quad -\infty < x < +\infty. \tag{4}$$

The condition of symmetry is given by

$$\frac{\partial \phi_1}{\partial y} = 0, \quad \text{on } y = 0, \quad -\infty < x < +\infty. \tag{5}$$

Since $P_1 - P_2$ is small, it follows from the gas laws that

$$\frac{\rho_2}{\rho_1} \simeq \frac{\gamma_2 a_1^2}{\gamma_1 a_2^2} = \frac{W_1^2 \ \gamma_2 \ M_2^2}{W_2^2 \ \gamma_1 \ M_1^2}$$

where a_1 and a_2 are the velocities of sound in the undisturbed regions of the jet and stream respectively. This relationship may be written

$$\frac{\rho_1 W_1^2}{\rho_2 W_2^2} = \frac{\gamma_1 M_1^2}{\gamma_2 M_2^2} .$$

The pressure (P_j) and density (ρ_j) in the jet are related by Bernoulli's theorem which, with (1), and the approximations of the linearised theory, has the form

$$W_1^2 \frac{\partial \phi_1}{\partial x} + \frac{\gamma_1}{\gamma_1 - 1} \frac{P_j}{\rho_j} = \frac{a_1^2}{\gamma_1 - 1} .$$

With the gas law this may be written

$$\left(\frac{P_j}{P_1}\right)^{\frac{\gamma_1 - 1}{\gamma_1}} = 1 - (\gamma_1 - 1) \frac{W_1^2}{a_1^2} \frac{\partial \phi_1}{\partial x} ,$$

and hence

$$\frac{P_j}{P_1} \simeq 1 - \frac{\gamma_1 W_1^2}{a_1^2} \frac{\partial \phi_1}{\partial x} .$$

Thus, the pressure in the jet is

$$P_j = P_1 - \rho_1 W_1^2 \frac{\partial \phi_1}{\partial x}$$

while, from a similar argument, the pressure (P_s) in the stream is

$$P_s = P_2 - \rho_2 W_2^2 \frac{\partial \phi_2}{\partial x}$$

to the first order of small quantities.

The condition that there is continuity of pressure ($P_j = P_s$) across the jet boundary ($y = h$) is given by

$$\frac{\partial \phi_2}{\partial x} - \frac{\gamma_1 M_1^2}{\gamma_2 M_2^2} \frac{\partial \phi_1}{\partial x} = -\varepsilon \,, \qquad y = h, \tag{6}$$

where $\varepsilon = \dfrac{(P_1 - P_2)}{\rho_2 W_2^2}$.

This form of the condition was used by Pack[15].

The differential equations for the perturbation velocities ϕ_1 and ϕ_2 are given, in the linearised form[20], by

$$\frac{\partial^2 \phi_1}{\partial y^2} - \beta_1^2 \frac{\partial^2 \phi_1}{\partial x^2} = 0 \tag{7}$$

and

$$\frac{\partial^2 \phi_2}{\partial y^2} + \beta_2^2 \frac{\partial^2 \phi_2}{\partial x^2} = 0, \tag{8}$$

where $\beta_1^2 = M_1^2 - 1$ and $\beta_2^2 = 1 - M_2^2$,

In order to apply the Wiener-Hopf technique to the problem it is convenient to rewrite the boundary conditions (2) to (6) and the field equations (7) and (8) in a more compact form. In the following work the suffixes 1 and 2 will, therefore, be retained only where necessary. Write

$$\phi(x,y) = \phi_+(x,y) + \phi_-(x,y)$$

where the functions $\phi_+(x,y)$ and $\phi_-(x,y)$ are defined by

$$\phi_+(x,y) = \begin{cases} \phi(x,y) & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and

$$\phi_-(x,y) = \begin{cases} 0 & \text{if } x > 0 \\ \phi(x,y) & \text{if } x < 0 \end{cases} .$$

The boundary conditions are then written

$$\frac{\partial \phi_+(x, h+0)}{\partial y} = \frac{\partial \phi_+(x, h-0)}{\partial y} \quad ; \tag{9}$$

$$\frac{\partial \phi_-(x, h+0)}{\partial y} = \frac{\partial \phi_-(x, h-0)}{\partial y} = 0 \quad ; \tag{10}$$

$$\frac{\partial \phi(x, 0)}{\partial y} = 0 \quad ; \tag{11}$$

$$\frac{\partial \phi(x, H-0)}{\partial y} = 0 \quad ; \tag{12}$$

and

$$\frac{\partial \phi_+(x, h+0)}{\partial x} - \ell^2 \frac{\partial \phi_+(x, h-0)}{\partial x} = -\varepsilon \quad , \tag{13}$$

where

$$\ell^2 = \frac{\gamma_1 M_1^2}{\gamma_2 M_2^2} .$$

The field equations are written

$$\frac{\partial^2 \phi}{\partial y^2} - \beta_1^2 \frac{\partial^2 \phi}{\partial x^2} = 0 ; \; 0 \leq y \leq h-0 , \; -\infty < x < +\infty \qquad (14)$$

and
$$\frac{\partial^2 \phi}{\partial y^2} + \beta_2^2 \frac{\partial^2 \phi}{\partial x^2} = 0 ; \; h+0 \leq y \leq H-0 , \; -\infty < x < +\infty \qquad (15)$$

The first step in the solution of the problem is to apply the x-Fourier transform theorem to the equations (9) to (15). The transform of the function $\phi(x,y)$ is written $\bar{\phi}(\alpha, y)$ where

$$\bar{\phi}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\alpha x} \phi(x,y) \, dx$$

and then
$$\bar{\phi}_+(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{i\alpha x} \phi_+(x,y) \, dx ,$$

$$\bar{\phi}_-(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\alpha x} \phi_-(x,y) .$$

It is necessary to examine the convergence of the Fourier integrals and this will be undertaken in the discussion of the problem. For the present, the convergence is assumed and the Fourier transform theorem applied to the various equations. The field equation(14) is transformed into the equation

$$\frac{d^2 \bar{\phi}(\alpha, y)}{d y^2} + \alpha^2 \beta_1^2 \bar{\phi} = 0 \qquad (16)$$

The solution of the transformed equation(16) is

$$\bar{\phi}(\alpha, y) = A \cos \alpha\beta_1 y + B \sin \alpha\beta_1 y \qquad (17)$$

where A and B are functions of α only.

When the transform is applied to the boundary condition (11) it gives

$$\frac{d\bar{\Phi}(\alpha,0)}{dy} = 0 \tag{18}$$

and it follows that the function B is zero, so that the solution for $\bar{\Phi}(\alpha, y)$, in the supersonic region, has the form

$$\bar{\Phi}(\alpha,y) = \bar{\Phi}_+(\alpha,y) + \bar{\Phi}_-(\alpha,y) = A(\alpha)\cos\alpha\beta_1 y \tag{19}$$

where $0 \leq y \leq h-0$.

In like manner the transform theorem may be applied to the field equation (15) and the boundary condition (12) to give the equations

$$\frac{d^2\bar{\Phi}(\alpha,y)}{dy^2} - \alpha^2\beta_2^2\,\bar{\Phi}(\alpha,y) = 0, \tag{20}$$

$$\frac{d\bar{\Phi}}{dy}(\alpha, H-0) = 0. \tag{21}$$

The appropriate solution for $\bar{\Phi}$ in the subsonic region has the form

$$\bar{\Phi}(\alpha,y) = \bar{\Phi}_+(\alpha,y) + \bar{\Phi}_-(\alpha,y) = C(\alpha)\cosh\alpha\beta_2(H-y) \tag{22}$$

where $h+0 \leq y \leq H-0$.

The Fourier transform theorem is applied to the boundary conditions (9) and (10) to yield the following equations:

$$\frac{d\bar{\Phi}_+(\alpha,h+0)}{dy} = \frac{d\bar{\Phi}_+(\alpha,h-0)}{dy} = \frac{d\bar{\Phi}_+(\alpha,h)}{dy}, \tag{23}$$

$$\frac{d\bar{\Phi}_-(\alpha,h+0)}{dy} = \frac{d\bar{\Phi}_-(\alpha,h-0)}{dy} = 0. \tag{24}$$

The transform theorem applied to (13) gives the result

$$\bar{\phi}_+(\alpha, h+0) - \ell^2 \bar{\phi}_+(\alpha, h-0) = \frac{\varepsilon}{\alpha^2\sqrt{2\pi}} - \frac{[\phi_+(0,h+0) - \ell^2 \phi_+(0,h-0)]}{i\,\alpha\sqrt{2\pi}}$$

The perturbation potentials ϕ_1 and ϕ_2 may be chosen such that $\phi_+(0,h+0)$ and $\phi_+(0,h-0)$ are both zero. If this is done the above equation becomes

$$\bar{\phi}_+(\alpha, h+0) - \ell^2 \bar{\phi}_+(\alpha, h-0) = \frac{\varepsilon}{\alpha^2\sqrt{2\pi}} \tag{25}$$

The boundary conditions (23) and (24) are used first in (19) to determine $A(\alpha)$ and to give the solution

$$\bar{\phi}(\alpha, y) = \bar{\phi}_+(\alpha, y) + \bar{\phi}_-(\alpha, y)$$

$$= -\frac{d\bar{\phi}_+(\alpha,h)}{dy} \cos\alpha\beta_1 y \Big/ \alpha\beta_1 \sin\alpha\beta_1 h \tag{26}$$

where $0 \leq y \leq h-0$, and second, in (22) to determine $C(\alpha)$ and to give the solution

$$\bar{\phi}(\alpha,y) = \bar{\phi}_+(\alpha,y) + \bar{\phi}_-(\alpha,y)$$

$$= -\frac{d\phi_+(\alpha,h)}{dy} \cosh[\alpha\beta_2(H-y)] \Big/ \alpha\beta_2 \sinh[\alpha\beta_2(H-h)] \tag{27}$$

where $h+0 \leq y \leq H-0$.

The equations (26) and (27) now satisfy the final boundary condition (25) and give

$$\frac{d\bar{\phi}_+(\alpha,h)}{dy}\left[\frac{\coth\{\alpha\beta_2(H-h)\}}{\beta_2} - \frac{\ell^2\cot\alpha\beta_1 h}{\beta_1}\right]$$

$$= -\frac{\varepsilon}{\alpha\sqrt{2\pi}} - \alpha\left[\bar{\phi}_-(\alpha,h+0) - \ell^2\bar{\phi}_-(\alpha,h-0)\right]$$

This may be re-written

$$\frac{\bar{v}_+(\alpha,h)\,K(\alpha)}{\beta_2} = -\frac{\varepsilon}{\alpha\sqrt{2\pi}} - \bar{G}_-(\alpha,h) \qquad , \tag{28}$$

where $$K(\alpha) = \coth\alpha\beta_2 L - m\coth\alpha\beta_1 h\,, \tag{29}$$

$$m = \ell^2\frac{\beta_2}{\beta_1} = \frac{\gamma_1 M_1^2\beta_2}{\gamma_2 M_2^2\beta_1} \qquad , \tag{30}$$

$$L = H - h \qquad , \tag{31}$$

$$\bar{v}_+(\alpha,h) = \frac{d\bar{\Phi}_+(\alpha,h)}{dy} \qquad , \tag{32}$$

and $$\bar{G}_-(\alpha,h) = \alpha\left[\bar{\Phi}_-(\alpha,h+0) - \ell^2\bar{\Phi}_-(\alpha,h-0)\right]. \tag{33}$$

The equation (28) is of the Wiener-Hopf kind and it is necessary to determine whether the terms can be rearranged in such a manner that one side of the equation is analytic in an upper half of the complex α-plane, whilst the other side is analytic in an overlapping lower half-plane and both sides analytic in the common strip. This rearrangement may be performed after investigation of the regions of analyticity of the functions $\bar{v}_+(\alpha,h)$; $K(\alpha)$; $1/\alpha$ and $\bar{G}_-(\alpha,h)$ of equation (28).

The basic theorem used in this investigation is the following (19 Pg 23) : If f(x) is a continuous function of x and $|f(x)| \to Ae^{\mu_- x}$ as $x \to +\infty$, then $\bar{F}_+(\alpha) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{i\alpha x} f(x)\,dx$ is regular for $\mathrm{Im}\,\alpha > \mu_-$. If

$|f(x)| \to B e^{\mu_+ x}$ as $x \to -\infty$, then $\bar{f}_-(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\alpha x} f(x)\, dx$ is regular for $\mathcal{I}m\, \alpha < \mu_+$.

The transform $\bar{v}_+(\alpha, h)$ is given by

$$\bar{v}_+(\alpha, h) = \frac{d\bar{\phi}_+(\alpha, h)}{dy} = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{0}^{\infty} e^{i\alpha x} \frac{\partial \phi_+(x, h)}{\partial y}\, dx$$

The perturbation velocity, and hence $\partial\phi_+(x, h)/\partial y$, must be bounded as $x \to +\infty$ and, if it is assumed integrable for a finite range of values of x it is integrable for an infinite range of values and it follows that $\bar{v}_+(\alpha, h)$ exists and is analytic in a region $\mathcal{I}m\, \alpha > 0$.

Next, the function $G_-(\alpha, h)$ is given by

$$\frac{G_-(\alpha, h)}{i} = -i\alpha\left[\bar{\phi}_-(\alpha, h+0) - \ell^2 \bar{\phi}_-(\alpha, h-0)\right]$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\alpha x} \frac{\partial \phi_-(x, h+0)}{\partial x}\, dx - \frac{\ell^2}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\alpha x} \frac{\partial \phi_-(x, h-0)}{\partial x}\, dx$$
$$- \frac{1}{\sqrt{2\pi}}\left[\phi_-(0, h+0) - \ell^2 \phi_-(0, h-0)\right]$$

The last term is zero if the perturbation potentials are chosen so that $\phi_-(0, h+0)$ and $\phi_-(0, h-0)$ are both zero. This choice, together with the fact that $\phi_+(0, h+0)$ and $\phi_+(0, h-0)$ are already chosen as zero, means that $\phi_1(0, h)$

and $\phi_2(0,h)$ are zero.

Due to the nature of supersonic flow, $\frac{\partial \phi_-}{\partial x}(x, h-0)$ is zero and the second integral in $G_-(\alpha, h)$ vanishes. Further, in the region $h+0 \leq y \leq H-0$, as $x \to -\infty$ then $\phi_-(x,y)$ is asymptotic to $\sum A_n e^{n\pi x/\beta_2 L} \cos[n\pi(H-y)/L]$. This is tantamount to the statement that the asymptotic form is the bounded solution of (15) which would have been obtained, as $x \to -\infty$, if the rigid wall at y = h extended to positive infinity. The function $G_-(\alpha, h)$ therefore exists and defines a function of α which is regular in the half plane $\mathscr{I}m\,\alpha < \pi/\beta_2 L$.

Next the function $1/\alpha$, given by $-\frac{i}{\sqrt{2\pi}}\int_0^\infty e^{i\alpha x}\,dx$, is analytic in the region $\mathscr{I}m\,\alpha > 0$ (there being a simple pole at $\alpha = 0$).

Finally there remains the discussion of the function $K(\alpha)$ defined by (29). It is evident that this function has poles on both the real and imaginary axes, the latter being situated at the zeros of $\tanh \alpha\beta_2 L = 0$. The function $K(\alpha)$ is certainly analytic in the region $0 < \mathscr{I}m\,\alpha < \pi/\beta_2 L$. Suppose now that $K(\alpha)$ can be factorised and written $K(\alpha) = K_+(\alpha)/K_-(\alpha)$ where $K_+(\alpha)$ is analytic and free of zeros in the upper half plane $\mathscr{I}m\,\alpha > 0$ and $K_-(\alpha)$ is analytic and free of zeros in the lower half plane $\mathscr{I}m\,\alpha < \rho \leq \pi/\beta_2 L$ where ρ is a positive real constant to be determined later. (See after equation (58).)

With this supposition the equation (28) may be rewritten

$$\frac{\bar{\vartheta}_+(\alpha,h)\,K_+(\alpha)}{\beta_2} = -\frac{\varepsilon K_-(\alpha)}{\alpha\sqrt{2\pi}} - G_-(\alpha,h)\,K_-(\alpha,h) \tag{34}$$

The left side of equation (34) is analytic in the upper half plane, the second term on the right side is analytic in the overlapping lower half plane and the first term on the right side is analytic in the common strip. However, this term may be decomposed into two terms each analytic in the appropriate half plane by writing

$$\frac{K_-(\alpha)}{\alpha} = \frac{K_-(0)}{\alpha} + \frac{[K_-(\alpha) - K_-(0)]}{\alpha} \tag{35}$$

In (35), the first term on the right side contains a pole at $\alpha = 0$ and is regular in the region $\mathrm{Im}\,\alpha > 0$ whilst the second term on the right side does not contain a pole and is regular in the lower half plane $\mathrm{Im}\,\alpha < \rho$.

Combine (34) with (35) and rearrange to give

$$\frac{\bar{\vartheta}_+(\alpha,h)\,K_+(\alpha)}{\beta_2} + \frac{\varepsilon K_-(0)}{\alpha\sqrt{2\pi}}$$

$$= -\frac{\varepsilon[K_-(\alpha) - K_-(0)]}{\alpha\sqrt{2\pi}} - G_-(\alpha,h)\,K_-(\alpha,h) \tag{36}$$

The left side of equation (36) is analytic in the upper half plane $\mathrm{Im}\,\alpha > 0$ the right side is analytic in the lower half plane $\mathrm{Im}\,\alpha < \rho$ and both sides are analytic in the common strip. Thus it follows that one side of (36) is the analytical continuation of the other. The two sides of (36) are therefore both equal to an integral function $E(\alpha)$. This gives two equations, namely

$$\frac{\bar{\vartheta}_+(\alpha, h)}{\beta_2} = -\frac{\varepsilon\, K_-(0)}{(\sqrt{2\pi})\,\alpha\, K_+(\alpha)} + \frac{E(\alpha)}{K_+(\alpha)} \tag{37}$$

and

$$G_-(\alpha, h) = -\frac{\varepsilon\,[K_-(\alpha) - K_-(0)]}{(\sqrt{2\pi})\,\alpha\, K_+(\alpha)} - \frac{E(\alpha)}{K_+(\alpha)} \tag{38}$$

Once the integral function $E(\alpha)$ has been determined these two equations may be used to find the inverse transforms of $\bar{\vartheta}_+(\alpha, h)$ and $G_-(\alpha, h)$ and, in fact, to find the perturbation potentials in the different regions of flow of the problem.

The formal evaluation of the function $E(\alpha)$ is carried out by investigating the order of growth, as $|\alpha| \to \infty$ in the appropriate half plane, of each side of equation (36) and then making appeal to an extended form of Liouville's theorem, it being arranged that both sides of (36) are of algebraical growth at infinity. However, without this formal process, it is possible to show that the integral function $E(\alpha)$ must be zero. For, from equation (6), it is noted that ε is zero when $P_1 = P_2$ and for this case there will be neither contraction nor expansion of the jet boundary and so

$\dfrac{\partial \phi(x,h)}{\partial y}$, $0 < x < \infty$ will be zero. This implies that

$\bar{\vartheta}_+(\alpha,h)$ is also zero and thus, from (37), it is seen that $E(\alpha)$ must vanish. This is formally shown to be true in Appendix 2.

With $E(\alpha)$ taken as zero (37) reduces to

$$\frac{\bar{\vartheta}_+(\alpha,h)}{\beta_2} = -\frac{\varepsilon\, K_-(0)}{(\sqrt{2\pi})\, \alpha\, K_+(\alpha)} \tag{39}$$

where $\dfrac{K_+(\alpha)}{K_-(\alpha)} = \coth \alpha\beta_2 L - m \cot \alpha\beta_1 h$

In the subsequent analysis, the equation (39) is used to determine the perturbation potentials. It may be noted that equation (38) could also be used to determine the potentials, but that it is sufficient to deal only with (39). In addition to the perturbation potentials it is useful to examine the displacement of the jet boundary. The equation of the jet boundary is written in the form

$$y = h[1 + f(x)], \quad f(x) = O(\varepsilon), \quad 0 < x < \infty. \tag{40}$$

The gradient of the boundary is thus $h\, f'(x)$ and hence

$$h\, f'(x) = \frac{\partial \phi_+(x,h)}{\partial y} \Big/ \left[1 + \frac{\partial \phi_+(x,h)}{\partial x}\right]$$

$$= \frac{\partial \phi_+(x,h)}{\partial y} \tag{41}$$

using the approximation of the linearised theory. The Fourier transform theorem may be applied to (41) to give the transform $\bar{f}(\alpha)$, of the function $f(x)$, in the form

$$-i\alpha h\bar{f}(\alpha) = \frac{\partial \bar{\phi}_+(\alpha, h)}{\partial y} = \bar{v}_+(\alpha, h) \qquad (42)$$

The working is simplified by a change of variable and to this end the following transformations are introduced

$$\xi = \alpha\beta_1 h \quad , \quad k = \beta_2 L/\beta_1 h \quad ,$$
$$t = x/\beta_1 h \quad , \quad v_+(x,y) = \varepsilon\beta_2 V_+(t,y)/\sqrt{2\pi} \quad , \qquad (43)$$

and

$$f(x) = i\varepsilon\beta_1\beta_2 F(t)/\sqrt{2\pi}$$
$$\phi(x,y) = \varepsilon h\beta_2\psi(t,y)/\sqrt{2\pi} \qquad (44)$$

In (44), the function $\psi(t,y)$ will frequently be called the perturbation potential.

It follows that

$$\bar{v}_+(\alpha, y) = \varepsilon h\beta_1\beta_2\bar{V}_+(\xi, y)/\sqrt{2\pi} \; , \qquad (45)$$

and

$$\bar{f}(\alpha) = i\varepsilon\beta_1^2\beta_2 h\bar{F}(\xi)/\sqrt{2\pi} \; , \qquad (46)$$

where $\bar{V}_+(\xi, h) = -Q_-(0)/\xi Q_+(\xi)$, (47)

$$\bar{F}(\xi) = -Q_-(0)/\xi^2 Q_+(\xi) , \tag{48}$$

and $Q(\xi) = Q_+(\xi)/Q_-(\xi) = \coth k\xi - m\cot\xi$ (49)

and the strip of analyticity of $Q(\xi)$ in the ξ-plane is $0 < \Im m\,\xi < \rho'$, where ρ' $(= \rho\beta_1 h)$ is a real positive constant yet to be determined.

It is necessary to investigate the factorisation of the function $Q(\xi)$. Now,

$$Q(\xi) = Q_+(\xi)/Q_-(\xi)$$

$$= \left[\tan\xi - m\tanh k\xi\right] / \left[\tan\xi \tanh k\xi\right]$$

and it may be shown that all the zeros of the numerator $[\tan\xi - m\tanh k\xi]$ lie on the real and imaginary axes of the ξ-plane (see Appendix 1). Write $\xi = \sigma + i\tau$. The real roots of $\tan\xi - m\tanh k\xi = 0$ are given by $\xi = 0$ and $\xi = \pm\sigma_n$ where $n\pi < \sigma_n < (2n+1)\pi/2$ and n = 0,1,2... if $mk > 1$ and n = 1,2... if $mk < 1$. The imaginary roots of $\tan\xi - m\tanh k\xi = 0$ are given by $\xi = 0$ and $\xi = \pm i\tau_n$, where $n\pi < k\tau_n < (2n+1)\pi/2$ and n = 1,2,3... if $mk > 1$ and n = 0,1,2,3... if $mk < 1$. In the critical case when mk = 1 the function $[\tan\xi - m\tanh k\xi]$ has a triple zero at the origin and the other zeros are given

by $\zeta = \pm\sigma_n$ and $\zeta = \pm i\tau_n$ with n = 1,2,3....
It may be noted (for all values of mk) that for large values of n

$$\sigma_n \sim n\pi + \theta,$$

and

$$k\tau_n \sim n\pi + \phi,$$

where $\theta = \tan^{-1} m$; $\phi = \tan^{-1}(1/m)$ and so $\theta + \phi = \frac{\pi}{2}$.

The zeros of $\tan\zeta$ and $\tanh k\zeta$ are $\zeta = \pm n\pi$ and $k\zeta = \pm i n\pi$ respectively, where n = 0,1,2... It follows that all the poles and zeros of $Q(\zeta)$ are known and that $Q(\zeta)$ may be expanded in an infinite product form by use of the factor theorem of Weierstrass (21). The expansion is different according as $mk > 1$, $mk = 1$ or $mk < 1$.

Consider first the case of $mk > 1$. The function $Q(\zeta)$ is written

$$Q(\zeta) = \frac{(1-mk)}{k\zeta}\left(1 - \zeta^2/\sigma_0^2\right)\prod_1^\infty \frac{\left(1 + \frac{\zeta^2}{\tau_n^2}\right)}{\left(1 + \frac{k^2\zeta^2}{n^2\pi^2}\right)}\;\frac{\left(1 - \frac{\zeta^2}{\sigma_n^2}\right)}{\left(1 - \frac{\zeta^2}{n^2\pi^2}\right)} \qquad (50)$$

and the factorisation into the form $Q_+(\zeta)/Q_-(\zeta)$ gives the following results;

$$Q_+(\zeta) = \frac{(1-mk)}{k\zeta}\left(1 - \frac{\zeta^2}{\sigma_0^2}\right)\prod_1^\infty \frac{\left(1 - \frac{i\zeta}{\tau_n}\right)e^{\frac{ik\zeta}{n\pi}}\left(1 - \frac{\zeta^2}{\sigma_n^2}\right)}{\left(1 - \frac{ik\zeta}{n\pi}\right)e^{\frac{ik\zeta}{n\pi}}\left(1 - \frac{\zeta^2}{n^2\pi^2}\right)} \qquad (51)$$

and $$Q_-(\xi) = \prod_1^\infty \frac{\left(1+\frac{ik\xi}{n\pi}\right)e^{-\frac{ik\xi}{n\pi}}}{\left(1+\frac{i\xi}{\tau_n}\right)e^{-\frac{i\xi k}{n\pi}}} \tag{52}$$

The insertion of the exponential terms in (51) and (52) ~~are~~ is necessary to ensure the absolute convergence of the product functions (21 Pg 32). The choice of these terms is not unique and, in fact, the natural choice might well have been, {for $Q_-(\xi)$ say,}

$$Q_-(\xi) = \prod_1^\infty \frac{\left(1+\frac{ik\xi}{n\pi}\right)e^{-\frac{ik\xi}{n\pi}}}{\left(1+\frac{i\xi}{\tau_n}\right)e^{-\frac{i\xi}{\tau_n}}}$$

If $Q_-(\xi)$ had been written thus, then it would be found to be of exponential order as $|\xi| \to \infty$ instead of the required algebraical order. This would not create any major difficulty since an exponential term could always be inserted into $Q_-(\xi)$ {and into $Q_+(\xi)$} on factorising $Q(\xi)$. The functions would, in fact, then be written

$$Q_+(\xi) = \frac{(1-mk)}{k\xi}\left(1-\frac{\xi^2}{\sigma_0^2}\right)e^{\chi(\xi)}\prod_1^\infty \frac{\left(1-\frac{i\xi}{\tau_n}\right)e^{\frac{i\xi}{\tau_n}}\left(1-\frac{\xi^2}{\sigma_n^2}\right)}{\left(1-\frac{ik\xi}{n\pi}\right)e^{\frac{ik\xi}{n\pi}}\left(1-\frac{\xi^2}{n^2\pi^2}\right)}$$

and $$Q_-(\xi) = e^{\chi(\xi)} \prod_1^\infty \frac{\left(1+\frac{ik\xi}{n\pi}\right) e^{-\frac{ik\xi}{n\pi}}}{\left(1+\frac{i\xi}{\tau_n}\right) e^{-\frac{i\xi}{\tau_n}}}$$

The unknown function $\chi(\xi)$ would then be determined by using the fact that $Q_\pm(\xi)$ are to be of algebraic growth at infinity. The introduction of $\chi(\xi)$ is avoided by using the exponential functions given in equations (51) and (52). The factorisation of $Q(\xi)$ has been formed in such a way that the function $Q_+(\xi)$ is analytic (and free of zeros) in the upper half plane $\Im m\, \xi > 0$ and the function $Q_-(\xi)$ is analytic (and free of zeros) in the overlapping half plane $\Im m\, \xi < \pi/k$.

Consider second the case $mk < 1$. In this case the factor theorem leads to

$$Q(\xi) = \frac{(1-mk)\left(1+\frac{\xi^2}{\tau_0^2}\right)}{k\xi} \prod_1^\infty \frac{\left(1+\frac{\xi^2}{\tau_n^2}\right)\left(1-\frac{\xi^2}{\sigma_n^2}\right)}{\left(1+\frac{k^2\xi^2}{n^2\pi^2}\right)\left(1-\frac{\xi^2}{n^2\pi^2}\right)} \tag{53}$$

so that the factorisation is in the form

$$Q_+(\xi) = \frac{(1-mk)\left(1-\frac{i\xi}{\tau_0}\right)}{k\xi} \prod_1^\infty \frac{\left(1-\frac{i\xi}{\tau_n}\right) e^{\frac{ik\xi}{n\pi}} \left(1-\frac{\xi^2}{\sigma_n^2}\right)}{\left(1-\frac{ik\xi}{n\pi}\right) e^{\frac{ik\xi}{n\pi}} \left(1-\frac{\xi^2}{n^2\pi^2}\right)} \tag{54}$$

and

$$Q_-(\xi)=\frac{1}{\left(1+\frac{i\xi}{\gamma_0}\right)}\prod_1^\infty \frac{\left(1+\frac{ik\xi}{n\pi}\right)e^{-\frac{ik\xi}{n\pi}}}{\left(1+\frac{i\xi}{\gamma_n}\right)e^{-\frac{ik\xi}{n\pi}}} \tag{55}$$

The function $Q_+(\xi)$ is analytic (and free of zeros) in the upper half plane $\mathfrak{Im}\,\xi > 0$ and $Q_-(\xi)$ is analytic (and free of zeros) in the lower half plane $\mathfrak{Im}\,\xi < \gamma_0$.

Consider last the case mk = 1 . The factor theorem leads to

$$Q(\xi)=\frac{(1+k^2)\,\xi}{3k}\prod_1^\infty \frac{\left(1+\frac{\xi^2}{\gamma_n^2}\right)\left(1-\frac{\xi^2}{\sigma_n^2}\right)}{\left(1+\frac{k^2\xi^2}{n^2\pi^2}\right)\left(1-\frac{\xi^2}{n^2\pi^2}\right)} \tag{56}$$

so that the factorisation is given by

$$Q_+(\xi)=\frac{(1+k^2)\,\xi}{3k}\prod_1^\infty \frac{\left(1-\frac{i\xi}{\gamma_n}\right)e^{\frac{ik\xi}{n\pi}}\left(1-\frac{\xi^2}{\sigma_n^2}\right)}{\left(1-\frac{ik\xi}{n\pi}\right)e^{\frac{ik\xi}{n\pi}}\left(1-\frac{\xi^2}{n^2\pi^2}\right)} \tag{57}$$

and

$$Q_-(\xi)=\prod_1^\infty \frac{\left(1+\frac{ik\xi}{n\pi}\right)e^{-\frac{ik\xi}{n\pi}}}{\left(1+\frac{i\xi}{\gamma_n}\right)e^{-\frac{i\xi k}{n\pi}}} \tag{58}$$

The function $Q_+(\xi)$ is analytic (and free of zeros) in the upper half plane $\mathfrak{Im}\,\xi > 0$ and $Q_-(\xi)$ is analytic (and free of zeros) in the lower half plane $\mathfrak{Im}\,\xi < \frac{\pi}{k}$.

For any value of mk it is noted that $Q_+(\xi)$ is analytic and non-zero in an upper half plane while $Q_-(\xi)$ is analytic and non-zero in an overlapping lower half plane and that $Q(\xi)$ is analytic in the common strip $0 < \mathcal{I}_m \xi < e'$ where it is sufficient to take $e' = \tau_0 \ (0 \leq \tau_0 < \pi/2k)$. This determines the value of the constant e' and hence of $e \ (= e'/\beta_1 h)$. The solution of the problem will be considered in each of the three cases $mk > 1$, $mk = 1$ and $mk < 1$.

The function $\bar{V}_+(\xi, h)$ defined by equation (47) may be written, by using (49), in the form

$$\bar{V}_+(\xi, h) = -\frac{1}{\xi(\coth k\xi - m\cot\xi)\ Q_-(\xi)} \tag{59}$$

where $Q_-(\xi)$ is given by equation (52) it being noted that $Q_-(0) = 1$.

In like manner the transform of the jet boundary displacement, given by (48), may be written

$$\bar{F}(\xi) = -\frac{1}{\xi^2(\coth k\xi - m\cot\xi)\ Q_-(\xi)} \tag{60}$$

The perturbation potential ϕ may be found as follows. From (43) the transform of $\phi(x,y)$ is given by

$$\bar{\phi}(\alpha, y) = \frac{\varepsilon h^2 \beta_1 \beta_2 \bar{\Psi}(\xi, y)}{\sqrt{2\pi}}$$

and thus, from (26), (32) and (45),

$$\bar{\psi}(\xi, y) = -\frac{\cos\left(\frac{\xi y}{h}\right)\ \bar{V}_+(\xi, h)}{\xi \sin \xi}$$

for the region $0 \leqslant y \leqslant h-0$.

The inverse transform theorem may be applied to yield

$$\psi(t, y) = -\frac{1}{\sqrt{2\pi}} \int_C \frac{e^{-i\xi t} \cos\left(\frac{\xi y}{h}\right)\ \bar{V}_+(\xi, h)\ d\xi}{\xi \sin \xi} \qquad (61)$$

$$0 \leq y \leq h-0,$$

where C is a path drawn from $-\infty$ to $+\infty$ in the strip $0 < \Im m\,\xi < \rho'$. Likewise use the equations (27), (32) and (45) to calculate

$$\bar{\psi}(\xi, y) = -\frac{\beta_1 \cosh[k\xi(H-y)/L]\ \bar{V}_+(\xi, h)}{\beta_2\, \xi \sinh k\xi}$$

and then

$$\psi(t, y) = -\frac{1}{\sqrt{2\pi}} \frac{\beta_1}{\beta_2} \int_C \frac{e^{-i\xi t} \cosh[k\xi(H-y)/L]\ \bar{V}_+(\xi, h)\ d\xi}{\xi \sinh k\xi} \qquad (62)$$

$$h+0 \leq y \leq H-0.$$

The path C of integration may be closed by an infinite semicircle in either the upper or lower half planes. The semicircle is drawn to pass between the poles of the integrands on both axes and it may be shown, due to the growth of $\bar{V}_+(\xi, h)$, that this is a valid closure of the contour and that the contribution from the semicircle is zero. Thus the integrals (61) and (62) may be evaluated using the Cauchy residue theorem.

Consider the supersonic region. The contour C of (61) may be closed by a semicircle in the upper half plane. Write $\xi = \sigma + i\tau = Re^{i\theta}$, then the modulus of the integrand of (61) is

$$e^{t\tau} \left| \frac{e^{\frac{i\sigma y}{h}} e^{-\frac{\tau y}{h}} + e^{-\frac{i\sigma y}{h}} e^{+\frac{\tau y}{h}}}{e^{i\sigma} e^{-\tau} + e^{-i\sigma} e^{\tau}} \right| \left| \bar{V}_+ \right| \quad d\theta$$

with $\tau = R\sin\theta$ and $\sin\theta > 0$

Let $|\xi| \to \infty$ then $|\bar{V}_+| = O\left(\frac{1}{R^{1+q}}\right) \to 0$ where $0 < q < \frac{1}{2}$ (see Appendix 2) and hence the integrand taken round the infinite semicircle vanishes if

$$e^{(t + \frac{y}{h} - 1)\tau} \to 0 \quad \text{as } \tau \to +\infty$$

that is, if $t + y/h - 1 < 0$.

Now the integrand of (61) has no poles in the region $\mathrm{Im}\,\xi > 0$ so that if the contour is closed in the upper half plane the contribution to the residue is zero. It follows that $\psi(t,y)$ (and hence $\phi(x,y)$) is zero in the region given by $t + y/h - 1 < 0$, $0 \leq y \leq h - 0$. In terms of the original variables, $\phi(x,y)$ is zero in the region $y - h + x/\beta_1 < 0$,

$0 \leqslant y \leqslant h-0$; and this shows that there is no disturbance in the supersonic jet upstream of the leading characteristic from the edge of the orifice. This result is as expected in a supersonic flow.

When the contour is closed by an infinite semicircle in the lower half plane, the function $\psi(t,y)$; $0 \leqslant y \leqslant h-0$; downstream of the leading characteristic is given by evaluating the residues at the poles of the integrand of (61) in this half plane.

Consider the subsonic region. The contour C may be closed by an infinite semicircle in the upper or in the lower half plane. Closure in the lower half plane provides the solution for $\psi(t,y)$ in the region $0 \leqslant t < +\infty$, $h+0 \leqslant y \leqslant H-0$, from the sum of the residues at the poles of the integrand of (62) in this half plane. Similarly the contribution to the solution of $\psi(t,y)$ in the region $-\infty < t \leqslant 0$, $h+0 \leqslant y \leqslant H-0$, comes from the sum of the residues at the poles on the positive imaginary axis of the ζ -plane, the path C being closed in the upper half plane. This latter contribution is not zero indicating that a disturbance has spread upstream in the subsonic region. Again, a result as expected on physical grounds.

Thus the function $\psi(t,y)$, and hence the perturbation potential $\phi(x,y)$, can be found in the form of an infinite series.

2. The solutions for the case $mk > 1$.

The problem of a jet in an infinite stream, discussed by Pai, is included in this case by letting k tend to infinity. The perturbation velocities in the various regions of flow and the fluctuations of the jet boundary are found in Sections 2.1, 2.2 and 2.3. Comparisons are drawn in 2.4 between the asymptotic form of the solution and the results obtained by Pai.

2.1 The supersonic region

The function $\overline{V}_+(\xi, h)$ defined by (59) has simple poles

$$\xi = \pm \sigma_0 \quad , \quad \xi = \pm \sigma_r \quad , \quad \xi = -i\tau_r$$

where r = 1,2,3... The denominator of the integrand (61) has a zero of order 2 at $\xi = 0$ and simple zeros at $\xi = \pm r\pi$ where r = 1,2,3....

If the contour of integration be closed in the lower half plane $\Im m\, \xi < \epsilon'$, the contour encloses the poles

$$\xi = 0 \;, \; \xi = \pm\sigma_0 ; \; \xi = \pm r\pi \;, \; \xi = -i\tau_r \;, \; \xi = \pm\sigma_r$$

where r = 1,2,3... and hence,

$$\psi(t, y) = \frac{1}{\sqrt{2\pi}} \int_C \frac{e^{-i\xi t} \cos(\xi y/h) \quad d\xi}{\xi^2 \sin\xi \left[\coth k\xi - m\cot\xi\right] \; Q_-(\xi)} \qquad (63)$$

in the region $t + y/h - 1 > 0, \; 0 \leq y \leq h - 0$.

This may be rewritten

$$\frac{i\,\psi(t,y)}{\sqrt{2\pi}} = R_1 + R_2 + R_3$$

where R_1 is the sum of the residues at the poles $\xi = 0, \pm r\pi,$

R_2 is the sum of the residues at the poles $\xi = \pm\sigma_0, \pm\sigma_r,$

R_3 is the sum of the residues at the poles $\xi = -i\tau_r$.

and $r = 1, 2, 3 \ldots$

The residue at the double pole $\xi = 0$ is the coefficient of the term ξ^{-1} in the expansion of the integrand of (63). If (52) be differentiated logarithmically it is found that

$$\frac{Q_-'(\xi)}{Q_-(\xi)} = \sum_1^\infty \left\{ \left[\frac{ik/n\pi}{(1 + ik\xi/n\pi)} - \frac{ik}{n\pi} \right] - \left[\frac{i/\tau_n}{(1 + i\xi/\tau_n)} - \frac{ik}{n\pi} \right] \right\}$$

and, since $Q_-(0) = 1$, it follows that

$$Q_-'(0) = \sum_1^\infty \frac{i(k\tau_n - n\pi)}{n\pi\tau_n}$$

and this is not zero. On recalling the fact that

$n\pi < k\tau_n < (2n+1)\pi/2$ it follows that

$$0 < Q_-'(0) < \sum_1^\infty \frac{ik}{2n^2\pi}$$

Finally, $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and so $Q_-'(0) = iC_1$ where $0 < C_1 < k\pi/12$. Further differentiation shows that $Q_-^{(r)}(0)$, where r = 2,3,4..., is non-zero.

The expansion of the integrand of (63) may then be written

$$\frac{[1 - i\xi t + O(\xi^2)]\ [1 + O(\xi^2)]}{\xi^2 [1 + O(\xi^2)][(\frac{1}{k} - m) + O(\xi^2)][1 + iC_1\xi + O(\xi^2)]}$$

The residue at the pole $\xi = 0$ is $-\frac{itk - iC_1 k}{(1 - mk)}$

The residue at the pole $\xi = r\pi$ is

$$e^{-ir\pi t} \cos(r\pi y/h)\ \bar{V}_+(r\pi, h) / r\pi \cos r\pi$$

which is zero due to the presence of the product function $\prod_{1}^{\infty} (1 - \xi^2/n^2\pi^2)$ in the numerator of $\bar{V}_+(\xi, h)$ (see equations (47) and (51)). The term R_1 is thus

$$-i\left[\frac{k(t + C_1)}{1 - mk}\right]$$

The term R_2 is given by

$$R_2 = \sum_{r=0}^{\infty} \left\{ \frac{\cos(\sigma_r y/h)}{\sigma_r^2 \sin \sigma_r} \frac{e^{i\sigma_r t}[Q_-(-\sigma_r)]^{-1} - e^{-i\sigma_r t}[Q_-(\sigma_r)]^{-1}}{[k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2 \sigma_r]} \right\}$$

$$= i \sum_{0}^{\infty} \frac{\cos(\sigma_r y/h)\,[A_r \sin \sigma_r t + B_r \cos \sigma_r t]}{\sigma_r^2 \sin \sigma_r \,[k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2 \sigma_r]}$$

and the term R_3 is given by

$$R_3 = -i \sum_{r=1}^{\infty} \frac{e^{-\tau_r t} \cosh(\tau_r y/h)\,[Q_-(-i\tau_r)]^{-1}}{\tau_r^2 \sinh \tau_r \,[k \operatorname{cosec}^2 k\tau_r - m \operatorname{cosech}^2 \tau_r]}$$

where $A_r = 2\,\mathrm{Re}\,[Q_-(-\sigma_r)]^{-1}$ and $B_r = 2\,\mathrm{Im}\,[Q_-(-\sigma_r)]^{-1}$ and $Q_-(\xi)$ is given by (52). It is noted that $Q_-(\xi)$ and $Q_-(-\xi)$ are complex conjugate functions.

In terms of the original spatial coordinates, x and y, the solution is given by adding the terms R_1 , R_2 and R_3 and applying the transformations (43) and (44). This leads to the equation

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = \frac{kx}{\beta_1 h (mk-1)} + \frac{C_1 k}{mk-1}$$

$$+ \sum_{r=0}^{\infty} \frac{\cos(\sigma_r y/h)\,[A_r \sin(x\sigma_r/\beta_1 h) + B_r \cos(x\sigma_r/\beta_1 h)]}{\sigma_r^2 \sin \sigma_r \,[k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2 \sigma_r]}$$

$$- \sum_{r=1}^{\infty} \frac{e^{-\tau_r t} \cosh(\tau_r y/h)\,[Q_-(-i\tau_r)]^{-1}}{\tau_r^2 \sinh \tau_r \,[k \operatorname{cosec}^2 k\tau_r - m \operatorname{cosech}^2 \tau_r]} \qquad (65)$$

in the region $0 \leq y \leq h-0$, $x/\beta_1 + y - h > 0$.

Since there are no poles in the upper half ξ -plane then

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = 0, \tag{66}$$

in the region $0 \leqslant y \leqslant h-0$, $x/\beta_1 + y - h < 0$.

2.2 <u>The subsonic region</u>

The integrand of equation (62) has poles in the lower half plane given at $\xi = 0$, $\xi = \pm\sigma_0$, $\xi = -\frac{ir\pi}{k}$, $\xi = -i\tau_r$ and $\xi = \pm\sigma_r$ and in the upper half plane at $\xi = +ir\pi/k$, where r = 1,2,3... Proceed as above for the supersonic region. With the contour closed in the lower half plane the residue at the pole $\xi = 0$ is $\left[\frac{it}{mk-1} + \frac{iC_1}{mk-1}\right]$; the residues at the poles $\xi = \pm\sigma_0$ and $\xi = \pm\sigma_r$ have a sum

$$i \sum_{r=0}^{\infty} \frac{\cosh[\sigma_r k(H-y)/L]}{\sigma_r^2 \sinh k\sigma_r} \left[\frac{A_r \sin(x\sigma_r/\beta_1 h) + B_r \cos(x\sigma_r/\beta_1 h)}{k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2 \sigma_r}\right]$$

Also, the residue at $\xi = -i\tau_r$ is

$$-i \sum_{r=1}^{\infty} \frac{e^{-\frac{x\tau_r}{\beta_1 h}} \cos[\tau_r k(H-y)/L]}{\tau_r^2 \sin k\tau_r} \left[\frac{\left[Q_-(-i\tau_r)\right]^{-1}}{k \operatorname{cosec}^2 k\tau_r - m \operatorname{cosech}^2 \tau_r}\right].$$

whilst the residue at $\xi = -ir\pi/k$ is zero because of the presence of the factor $(1 - ik\xi/n\pi)$ on the numerator of $\overline{V}_+(\xi, h)$ (see equations (47) and (51)).

With the closure of the contour in the upper half plane the contribution to the integral (62) arises from the poles at

$\zeta = +ir\pi/k$. The sum of the residues at these poles is

$$\sum_{1}^{\infty} \frac{e^{-r\pi t/k} \cos[r\pi(H-y)/L] \; \bar{V}_+(ir\pi/k, h)}{ir\pi \cos r\pi} .$$

Thus, in terms of the original variables the solution of the potential $\phi(x,y)$ in the subsonic region is

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} = \frac{x}{h\beta_1(mk-1)} + \frac{C_1}{(mk-1)}$$
$$+ \sum_{0}^{\infty} \frac{\cosh[\sigma_r k(H-y)/L]}{\sigma_r^2 \sinh k\sigma_r} \left[\frac{A_r \sin(x\sigma_r/\beta_1 h) + B_r \cos(x\sigma_r/\beta_1 h)}{k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2 \sigma_r} \right]$$
$$- \sum_{1}^{\infty} \frac{e^{-x\tau_r/\beta_1 h} \cos[\tau_r k(H-y)/L]}{\tau_r^2 \sin k\tau_r} \left[\frac{\{Q_-(-i\tau_r)\}^{-1}}{k \operatorname{cosec}^2 k\tau_r - m \operatorname{cosech}^2 \tau_r} \right] \qquad (67)$$

in the region $x > 0, \quad h+0 \leq y \leq H-0$, and

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} = \sum_{1}^{\infty} \frac{e^{-r\pi x/\beta_2 L} \cos[r\pi(H-y)/L] \; \bar{V}_+(ir\pi/k, h)}{r\pi \, (-1)^{r+1}} \qquad (68)$$

in the region $x<0, \; h+0 \leq y \leq H-0$.

2.3 <u>The behaviour of the jet boundary</u>

The behaviour of the jet boundary may be examined by calculating the function f(x) of equation (40). From (48) and (49)

$$\bar{F}(\xi) = -\frac{1}{\xi^2 Q_+(\xi)}$$

$$= -\frac{1}{\xi^2[\coth k\xi - m\cot\xi]\, Q_-(\xi)}$$

where $Q_-(\xi)$ is given by (52), in the case $mk > 1$, it being noted that $Q_-(0) = 1$.

It follows that

$$F(t) = -\frac{1}{\sqrt{2\pi}} \int_C \frac{e^{-i\xi t}\, d\xi}{\xi^2(\coth k\xi - m\cot\xi)\, Q_-(\xi)} \qquad (69)$$

in which the integrand has simple poles at $\xi = 0$, $\xi = \pm\sigma_0$, $\xi = \pm\sigma_r$ and $\xi = -i\tau_r$ where r = 1,2,3....

If it is required, the general solution of F(t), and hence of f(x), can be calculated by evaluating the residues at the poles of the integrand in (69). However, particular interest is attached to the initial slope of the jet boundary and to the ultimate width of the jet. These end solutions may now be considered.

The ultimate width of the jet is found by calculating the limiting value of f(x) as $x \to \infty$ and this is given by the asymptotic solution of (69) when the contour is closed in the lower half plane. The asymptotic form of F(t) is given by the contribution to the residues from the poles with the greatest imaginary part (22 Pg 279). These poles are simply

$\xi = 0$; $\xi = \pm\,\sigma_0$ and $\xi = \pm\,\sigma_r$.

Near $\xi = 0$ the integrand of (69) has the Taylor expansion

$$\frac{\left[1 - i\xi t + O(\xi^2)\right]}{\xi^2\left[\left(\frac{1}{k} - m\right)\frac{1}{\xi} + O(\xi)\right]\left[1 + iC\xi + O(\xi^2)\right]}$$

which may be rewritten

$$\frac{\left[1 - i\xi t + O(\xi^2)\right]\left[1 + O(\xi)\right]}{\left(\frac{1}{k} - m\right)\xi}\ .$$

If note be made of the sense of description of the contour it will be found that the contribution from the pole at the origin is

$$-\ \frac{i\,k\sqrt{2\pi}}{(mk - 1)}\ .$$

The contribution which arises from the poles $\xi = \pm\,\sigma_r$ $r = 0,1,2,3\ldots$ lead, as before, to terms in $\sin \sigma_r t$ and $\cos \sigma_r t$.

From (44), the asymptotic solution for f(x) is then given by

$$\frac{f(x)}{\varepsilon\,\beta_1\beta_2} \sim \frac{k}{(mk-1)} + \text{terms in} \left\{\begin{matrix}\sin \\ \cos\end{matrix}\right. \left(\frac{\sigma_r x}{\beta_1 h}\right) \qquad (70)$$

The fluctuations are not periodic because of the location of the roots σ_r , but when k is large the roots are very nearly given by $\left[r\pi + \tan^{-1} m\right]$ and the functions are then almost periodic (23). The conclusion drawn is that, far downstream, the width of the jet boundary fluctuates about a mean value $h[1 + \varepsilon\beta_1\beta_2 k/(mk-1)]$ and, when k is large, this approximates to $h[1 + \varepsilon\beta_1\beta_2/m]$. This indicates that in the over pressure case $(P_1 > P_2)$ when $\varepsilon > 0$, there is a small overall increase in the jet width and in the under-pressure case when $\varepsilon < 0$ there is an overall decrease in the jet width. Note that when both pressures P_1 and P_2 are the same there is no movement of the jet boundary.

The estimate of F(t) near the orifice is found by taking the inverse transform of the asymptotic form of $\bar{F}(\xi)$.

Now

$$\bar{F}(\xi) = -\frac{1}{\xi\ Q_+(\xi)}$$

and the asymptotic form for large $|\xi|$ may be obtained as is shown in Appendix 2. By using the Stirling expansion formulae for the various Gamma functions which appear in the expression $Q_+(\xi)$ it may be shown that

$$Q_+(\xi) \sim A\,\xi^q / e^{i\pi(q+1)/2}$$

and that

$$\bar{F}(\xi) \sim -e^{\frac{i\pi(q+1)}{2}} / A\,\xi^{q+2}$$

where $|\xi| \to \infty$, $\mathrm{Im}\,\xi < e'$ and A is real positive constant.
(24 Pg 129)
By a Tauberian type theorem of inverse transform theory the asymptotic form for F(t) as $t \to 0+$ can be implied from the asymptotic form of $\bar{F}(\xi)$ as $|\xi| \to \infty$.

In fact,

$$F(t) \sim \frac{e^{i\pi(q+1)/2} \quad \sqrt{2\pi} \quad t^{q+1}}{A\,\Gamma(q+2) \quad e^{i(q+2)\pi/2}}$$

after noting that $|\xi|$ tends to infinity in the lower half plane and taking the sense of description of the contour into account. Thus

$$F(t) \sim i\,B\,t^{q+1}$$

where B is a real positive constant. With use of the transformations (43) and (44) this is equivalent to

$$f(x) \sim \varepsilon\, C_2\, x^{q+1} \quad as \quad x \to 0+ \tag{71}$$

where C_2 is a positive real constant.

The number q is constant for any given value of the function $m\left(= \gamma_1 M_1^2 \beta_2 / \gamma_2 M_2^2 \beta_1\right)$ and lies in the range $0 < q < \frac{1}{2}$, thus, initially, (for $mk > 1$) the jet boundary expands (when $\varepsilon > 0$) and the slope of the boundary lies between the two limiting curves $f(x) = O(x)$ and $f(x) = O(x^{3/2})$. The slope of the jet is zero at the orifice and increases initially like the curve $y = O(x^{q+1})$.

2.4 <u>Comparison with the solution obtained by S.I. Pai</u>

At the beginning of the chapter it was observed that the solution to the problem of a supersonic jet emitting into an infinite subsonic stream was attempted by Pai (13), but that in formulating the problem certain boundary conditions were omitted. In consequence, the solution that he obtained could only be valid far downstream of the jet exit. It has not been found possible to start from the solutions of the present problem, namely equations (65) to (68), and proceed to the limit as $k \to \infty$ and so provide the correct solutions for the case of an infinite subsonic stream but, nevertheless, it may be shown that the solutions obtained by Pai are the asymptotic forms of the solutions (65) and (67).

Start with the equation (65) which gives the potential inside the supersonic jet downstream of the jet exit,

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = \frac{kx}{\beta_1 h(mk-1)} + \frac{C_1 k}{(mk-1)}$$

$$+\sum_{r=0}^{\infty} \frac{\cos(\sigma_r y/h)}{\sigma_r^2 \sin \sigma_r}\left[\frac{A_r \sin(x\sigma_r/\beta_1 h) + B_r \cos(x\sigma_r/\beta_1 h)}{k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2 \sigma_r}\right]$$

$$+\sum_{r=1}^{\infty} \frac{e^{-\tau_r x/\beta_1 h} \cosh(\tau_r y/h)}{\tau_r^2 \sinh \tau_r}\left[\frac{\{Q_-(-i\tau_r)\}^{-1}}{[k \operatorname{cosec}^2 k\tau_r - m \operatorname{cosech}^2 \tau_r]}\right]$$

$$0 \leq y \leq h-0, \quad x/\beta_1 + y - h > 0,$$

and consider the terms as $x \to +\infty$.

The first term tends to infinity with x but simply signifies a steady component, parallel to the jet axis, of the

perturbation velocity. The fourth term tends to zero due to the presence of the exponential term $e^{-\gamma_r x/\beta_1 h}$. The second term is constant and does not affect the state of flow, whilst the third term provides the oscillatory terms far downstream. It is recalled that σ_n (n = 0,1,2,3,...) are the real solutions of the equation $\tan\xi - m\tanh k\xi = 0$.
As k (and hence the outer stream width) tends to infinity the roots σ_n rapidly approximate to the roots of $\tan\xi = m$ and so $\sigma_n \to (n\pi + \Theta)$ where $\Theta = \tan^{-1} m$. Also, use may be made of the identity $\coth k\sigma_r = m\cot\sigma_r$ to give a reduction of the expression $[k\,\mathrm{cosech}^2 k\sigma_r - m\,\mathrm{cosec}^2\sigma_r]$ which appears in the above equation. For large values of k,

$$[k\,\mathrm{cosech}^2 k\sigma_r - m\,\mathrm{cosec}^2\sigma_r]$$
$$= k(m^2\cot^2\sigma_r - 1) - m(1+\cot^2\sigma_r)$$
$$= -(m + 1/m) \quad \text{since } \cot\sigma_r = 1/m.$$

It may be noted that this expression is independent of k.
Finally, as $x \to \infty$

$$\frac{\phi(x,y)}{\varepsilon h\beta_2} \sim \frac{x}{m\beta_1 h} + \sum_{r=0}^{\infty} \cos[(\theta+r\pi)y/h]\left[A_{1r}\cos[x(\theta+r\pi)/\beta_1 h] + B_{1r}\sin\left[\frac{x(\theta+r\pi)}{\beta_1 h}\right]\right] \quad (72)$$

where $0 < \Theta < \frac{\pi}{2}$ and A_{1r}, B_{1r} are coefficients dependent on A_r and B_r defined by (64).

The solution (72) is identical with the solution obtained by Pai except for the first term which does not appear in his solution. However, this is accounted for by the fact that in equation (1) the supersonic velocity W_1 is defined as the velocity in the undisturbed jet whereas the equivalent velocity in Pai's analysis is taken as the mean velocity in the jet where the pressure in the jet is equal to the pressure of the undisturbed uniform (outer) stream.

Let a_1 and a_2 be the speeds of sound in the jet and outer stream respectively. To a first approximation

$$\frac{\rho_2}{\rho_1} = \frac{\gamma_2 a_1^2}{\gamma_1 a_2^2} = \left[W_1^2/\gamma_1 M_1^2 \right] \Big/ \left[W_2^2/\gamma_2 M_2^2 \right]$$

and hence

$$\frac{\rho_2 W_2^2}{\rho_1 W_1^2} = \frac{\gamma_2 M_2^2}{\gamma_1 M_1^2}$$

Further, to the same approximation, the pressure P_j in the jet is given by

$$P_j = P_1 - \rho_1 W_1^2 \left(\frac{\partial \phi_1}{\partial x} \right)$$

and this is equal to the pressure P_2 of the undisturbed outer stream when

$$\frac{\partial \phi_1}{\partial x} = \frac{P_1 - P_2}{\rho_1 W_1^2} = \frac{\varepsilon \rho_2 W_2^2}{\rho_1 W_1^2} = \varepsilon \frac{\gamma_2 M_2^2}{\gamma_1 M_1^2} .$$

With equation (30) this leads to

$$\frac{\partial \phi_1}{\partial x} = \frac{\varepsilon \beta_2}{m \beta_1} \qquad (73)$$

This is simply the extra term in the differentiated form of equation (72). Thus the solution given by Pai (13 Eq 12) is the asymptotic form of the present solution as the outer stream width tends to infinity.

In like fashion an examination can be made of the asymptotic solution, for the case $k \to \infty$, of the perturbation potential in the subsonic stream and compared with the solution given by Pai.

Start with equation (67)

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} = \frac{x}{h\beta_1(mk-1)} + \frac{C_1}{(mk-1)}$$

$$+ \sum_{0}^{\infty} \frac{\cosh[\sigma_r k(H-y)/L]}{\sigma_r^2 \sinh k\sigma_r} \left[\frac{A_r \sin(x\sigma_r/\beta_1 h) + B_r \cos(x\sigma_r/\beta_1 h)}{k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2 \sigma_r} \right]$$

$$- \sum_{1}^{\infty} \frac{e^{-x\tau_r/\beta_1 h} \cos[\tau_r k(H-y)/L]}{\tau_r^2 \sin k\tau_r} \left[\frac{\{Q_-(-i\tau_r)\}^{-1}}{k \operatorname{cosec}^2 k\tau_r - m \operatorname{cosech}^2 \tau_r} \right]$$

$$h+0 \leq y \leq H-0,$$

and consider the terms as $x \to +\infty$.

As in the supersonic region, the first term indicates a constant steady component of the perturbation velocity, the fourth term tends exponentially to zero the second term is a constant whilst the third term leads to the form of solution as given by Pai.

Thus as $x \to \infty$

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} \sim \frac{x}{h\beta_1(mk-1)}$$

$$+ \sum_{0}^{\infty} \frac{\cosh[\sigma_r k(H-y)/L]}{\sigma_r^2 \sinh k\sigma_r} \left[\frac{A_r \sin(x\sigma_r/\beta_1 h) + B_r \cos(x\sigma_r/\beta_1 h)}{k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2 \sigma_r} \right]$$

Now let the outer stream width tend to infinity. This is achieved by making $H \to \infty$ whilst h is held fixed. Since $L = (H - h)$ and $k = \beta_2 L/\beta_1 h$ it follows that

$$\frac{\cosh[\sigma_r k(H-y)/L]}{\sinh k\sigma_r} = \frac{\cosh\left[\frac{\sigma_r \beta_2 (H-y)}{\beta_1 h}\right]}{\sinh\left[\frac{\sigma_r \beta_2 (H-h)}{\beta_1 h}\right]}$$

$$= \frac{e^{\lambda H} e^{-\lambda y} + e^{-\lambda H} e^{+\lambda y}}{e^{\lambda H} e^{-\lambda h} - e^{-\lambda H} e^{+\lambda h}}$$

where $\lambda = \sigma_r \beta_2/\beta_1 h$.

As H tends to infinity, the above expression in y tends towards $e^{-\lambda(y-h)}$.

As before, the term $[k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2 \sigma_r]$ is independent of k as $k \to \infty$ and $\sigma_r \sim r\pi + \theta$.

It may be noted that the steady component of the perturbation velocity in the stream is $\frac{1}{h\beta_1(mk-1)}$ and that this tends to zero as $k \to \infty$. Thus, for the case of an infinite subsonic stream, the solution far downstream $(x \to +\infty)$ is given by

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} \sim \sum_0^\infty e^{-\left[\left(\frac{\beta_2}{\beta_1 h}\right)(n\pi+\theta)y\right]} \left[A_{2n} \cos\left\{\frac{(n\pi+\theta)x}{\beta_1 h}\right\} + B_{2n} \sin\left\{\frac{(n\pi+\theta)x}{\beta_1 h}\right\}\right] \quad (74)$$

where A_{2n} and B_{2n} are constants dependent on A_n and B_n.

The solution (74) is identical with that given by Pai (13 Eqn.13). The solution far upstream, in the subsonic case, is provided by equation (68) and it is sufficient to observe that the solution vanishes exponentially as $x \to -\infty$.

Thus it is shown that the solutions given by Pai can only hold true far away from the jet exit the complete general solution being provided by equations (65)-(68) for the case $k \to \infty$.

3. <u>The solutions for the case $mk < 1$</u>.

This case provides a solution which differs from the preceding one in certain respects. The chief difference is due to the location of the roots of the expression $\tan\xi - m\tanh k\xi = 0$. This equation does not now have the real roots $\xi = \pm\sigma_0$ where $0 < \sigma_0 < \pi/2$ but has instead two imaginary roots $\xi = \pm i\tau_0$ where $0 < k\tau_0 < \pi/2$. This modifies the factorisation of $Q(\xi)$ and hence the representation of $Q_+(\xi)$ and $Q_-(\xi)$ which in turn leads to a different evaluation of the contour integrals. Again, the new factorisation alters the order of the functions $Q_+(\xi)$ and $Q_-(\xi)$ at infinity (see Appendix 2) and, in consequence, it will be seen that this amends the initial jet slope.

The modified potential function $\psi(t,y)$ is to be evaluated, as before, from the integrals (61) and (62), which may be rewritten

$$\psi(t,y) = \frac{1}{\sqrt{2\pi}} \int_C \frac{e^{-i\xi t} \quad \cos(\xi y/k) \quad d\xi}{\xi^2 \sin\xi \ (\coth k\xi - m\cot\xi) \quad Q_-(\xi)}$$

$$0 \leq y \leq k-0,$$

and

$$\psi(t,y) = \frac{1}{\sqrt{2\pi}} \; \frac{\beta_1}{\beta_2} \int_C \frac{e^{-i\zeta t} \quad \cosh\left[k\zeta(H-y)/L\right] \quad d\zeta}{\zeta^2 \; \sinh k\zeta \; (\coth k\zeta - m\cot\zeta) \quad Q_-(\zeta)}$$

$$h+0 \leq y \leq H-0,$$

where C is a path drawn from $-\infty$ to $+\infty$ in the strip of analyticity $0 < \Im m\,\zeta < e'(=\tau_0)$.

As in the previous case for $mk > 1$ it may be shown by closing the contour C, in (61), by an infinite semicircle in the upper half plane, that the function is zero upstream of the leading characteristic $t + y/h - 1 < 0$, $0 < y < h-0$, it being noted that, in this case, $\overline{V}_7(\zeta, h) = O(R^{-q})$, $0 < q < \frac{1}{2}$ where $|\zeta| = R \to \infty$.

3.1 <u>The supersonic region</u>

The solution in this region may be examined in some detail. The integrand of equation (61) has a double pole at $\zeta = 0$, and simple poles at $\zeta = \pm\sigma_r$; $\zeta = -i\tau_0$; $\zeta = -i\tau_r$ and $\zeta = \pm r\pi$ where r = 1,2,3... When the contour is closed by an infinite semicircle in the lower half plane the contributions to $\psi(t,y)$ arise from the residues of the integrand of (61) at these poles. Adopt the procedure of the previous case and write

$$i\psi(t,y)/\sqrt{2\pi} = R_1 + R_2 + R_3$$

where R_1 is the sum of the residues at the poles $\zeta = 0, \pm r\pi$,

R_2 is the sum of the residues at the poles $\zeta = \pm\sigma_r$, and R_3 is the sum of the residues at the poles $\zeta = -i\gamma_0, -i\tau_r$ where r = 1,2,3...

The residue at the double pole $\zeta = 0$ is the coefficient of term in ζ^{-1} in the Taylor's expansion of the integrand of (61). This residue differs from that of the similar term of the previous case because of the different way in which $Q_-(\zeta)$ is expressed. On differentiating (55) logarithmically it is found that

$$\frac{Q_-'(\zeta)}{Q_-(\zeta)} = -\frac{i/\gamma_0}{(1+i\zeta/\gamma_0)} + \sum_1^\infty \left\{ \left[\frac{ik/n\pi}{(1+ik\zeta/n\pi)} - \frac{ik}{n\pi} \right] - \left[\frac{i/\tau_n}{(1+i\zeta/\tau_n)} - \frac{ik}{n\pi} \right] \right\}$$

and hence, since $Q_-(0) = 0$, it follows that

$$Q_-'(0) = -\frac{i}{\gamma_0} + \sum_1^\infty \frac{i(k\gamma_n - n\pi)}{n\pi\,\gamma_n} = i\,B_1$$

where $B_1 = C_1 - 1/\gamma_0$ and $0 < C_1 < \pi k/12.$

The expansion of the integrand of equation (61) is then written

$$\frac{\left[1 - i\zeta t + o(\zeta^2)\right]\left[1 + o(\zeta^2)\right]\left[1 - iB_1\zeta + o(\zeta^2)\right]}{\zeta^2\left[\left(\frac{1}{k} - m\right) + o(\zeta^2)\right]}$$

The contribution to the residues from the poles $\zeta = \pm r\pi$ is still zero and so it follows that the term R_1 is

$$-\frac{ik\,(t + B_1)}{(1 - mk)}\quad .$$

The calculation of the terms R_2 and R_3 is exactly as before and leads to

$$R_2 = i \sum_{r=1}^{\infty} \frac{\cos(\sigma_r y/h)\left[A_r \sin\sigma_r t + B_r \cos\sigma_r t\right]}{\sigma_r^2 \sin\sigma_r \left[k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2\sigma_r\right]}$$

and

$$R_3 = -i \sum_{r=0}^{\infty} \frac{e^{-\tau_r t} \cosh(\tau_r y/h) \left[Q_-(-i\tau_r)\right]^{-1}}{\tau_r^2 \sinh\tau_r . \left[k \operatorname{cosec}^2 k\tau_r - m \operatorname{cosech}^2 \tau_r\right]}$$

Where $A_r = 2\,Re\left[Q_-(-\sigma_r)\right]^{-1}$ and $B_r = 2\,\Im m\left[Q_-(-\sigma_r)\right]^{-1}$
and $Q_-(\xi)$ is given by the equation (55).
The final expression for $\phi(x, y)$ in the region of the supersonic jet for the case $mk < 1$ is then given by adding the terms R_1, R_2 and R_3 and applying the transformation (43). Thus

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = -\frac{k(x + B_1\beta_1 h)}{\beta_1 h(1 - mk)}$$

$$+ \sum_{1}^{\infty} \frac{\cos(\sigma_r y/h)}{\sigma_r^2 \sin\sigma_r} \frac{\left[A_r \sin(x\sigma_r/\beta_1 h) + B_r \cos(x\sigma_r/\beta_1 h)\right.}{\left[k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2\sigma_r\right]}$$

$$- \sum_{r=0}^{\infty} \frac{e^{-\tau_r x/\beta_1 h} \cosh(\tau_r y/h) \left[Q_-(-i\tau_r)\right]^{-1}}{\tau_r^2 . \sinh\tau_r . \left[k \operatorname{cosec}^2 k\tau_r - m \operatorname{cosech}^2\tau_r\right]} \qquad (75)$$

in the region $0 \leq y \leq h-0$, $x/\beta_1 + y - h > 0$

and
$$\phi(x,y)/\varepsilon h \beta_2 = 0 \qquad (76)$$

in the region $0 \leq y \leq h-0$, $x/\beta_1 + y - h < 0$.

It may be noted that although the same notation is used in (75) as in (65), the actual position of the roots σ_r and $i\tau_r$ are different according as $mk > 1$, $mk = 1$ or $mk < 1$. In fact, if the roots were designated σ_r' , τ_r' for the case $mk > 1$; σ_r'', τ_r'' for the case mk = 1 ; and σ_r''', τ_r''' for the case $mk < 1$ it may easily be shown that $\sigma_r' > \sigma_r'' > \sigma_r'''$ and $\tau_r' < \tau_r'' < \tau_r'''$. However, for large values of r the roots σ_r and $k\tau_r$ are asymptotic to $r\pi + \tan^{-1} m$ and $r\pi + \tan^{-1}(1/m)$ respectively. For a theoretical discussion it is not necessary to retain the primes.

It may also be noted that the term $\dfrac{B_1 k}{\beta_1 h(1-mk)}$ which appears in (75) is a constant and does not affect the flow.

3.2 <u>The subsonic region</u>

In this region the solution is obtained from (62) in which the integrand has poles in the lower half plane at $\zeta = 0$, $\zeta = -ir\pi/k$, $\zeta = \pm\sigma_r$, $\zeta = -i\tau_0$ and $\zeta = -i\tau_r$; and in the upper half plane at $\zeta = +ir\pi/k$ where r = 1,2,3... Proceed as already indicated in the previous evaluations, and the solution is found to be

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} = -\frac{(x + B_1\beta_1 h)}{\beta_1 h (1-mk)}$$

$$+ \sum_{1}^{\infty} \frac{\cosh[\sigma_r k(H-y)/L]}{\sigma_r^2 \sinh k\sigma_r} \frac{[A_r \sin(x\sigma_r/\beta_1 h) + B_r \cos(x\sigma_r/\beta_1 h)]}{[k \operatorname{cosech}^2 k\sigma_r - m \operatorname{cosec}^2 \sigma_r]}$$

$$+ \sum_{0}^{\infty} \frac{e^{-x\tau_r/\beta_1 h} \cos[k\tau_r(H-y)/L] \; [Q_-(-i\tau_r)]^{-1}}{\tau_r^2 \sin k\tau_r \; [k \operatorname{cosec}^2 k\tau_r - m \operatorname{cosech}^2 \tau_r]}$$

in the region $x > 0, \; h+0 \leq y \leq H-0,$

and

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} = \sum_{1}^{\infty} \frac{e^{r\pi x/\beta_2 L} \cos[r\pi(H-y)/L] \; \bar{V}_r(\frac{ir\pi}{k}, h)}{r\pi(-1)^{r+1}}$$

in the region $x < 0, \; h+0 \leq y \leq H-0.$

It is observed that these solutions for the case $mk < 1$ differ from the previous solutions for $mk > 1$ only in the number of terms in the summations, the form of the solution remains the same.

3.3 The behaviour of the jet boundary

In order to examine the ultimate jet width it is necessary to find the limiting value, as $x \to +\infty$, of the function f(x) defined by equation (40). This function is obtained by examining the function $F(t)$ defined by equation (44) and given in integral form by equation (69). The integrand of (69) has a simple pole at $\zeta = 0$ and other poles at $\zeta = -i\gamma_0$; $-i\gamma_r$, $\pm\sigma_r$; r = 1,2,3... Near $\zeta = 0$ the Taylor expansion of the integrand takes the form

$$\frac{\left[1 - i\zeta t + o(\zeta^2)\right]\left[1 + o(\zeta^2)\right]\left[1 - iB_1\zeta + o(\zeta^4)\right]}{\zeta\left[\frac{1}{k} - m\right]}$$

and the contribution to $F(t)$ is thus $-\dfrac{ik\sqrt{2\pi}}{(mk-1)}$.

If account be taken of the contributions from the other poles $\pm\sigma_r$, the asymptotic solution of $f(x)$ is given by

$$\frac{f(x)}{\varepsilon\beta_1\beta_2} \sim -\frac{k}{(1-mk)} + \text{terms in} \begin{cases}\cos\\ \sin\end{cases}\left(\frac{\sigma_r x}{\beta_1 h}\right)$$

As in Section 2.3 the sine and cosine terms provide the fluctuations about the mean displacement. In this case the boundary fluctuates about a mean value of $h\left[1 - \frac{\varepsilon\beta_1\beta_2 k}{1-mk}\right]$. Note that here $mk < 1$ and so in the over-pressure case when $\varepsilon > 0$ there is an overall decrease in the jet width whilst in the under-pressure case there is an overall increase in jet width. These results are the opposite to those obtained for the case $mk > 1$.

The initial slope of the jet boundary is found by investigating the behaviour of the transform $\bar{F}(\xi)$ as $|\xi| \to \infty$. Now

$$\bar{F}(\xi) = -\frac{1}{\xi^2 Q_+(\xi)}$$

and the asymptotic form may be obtained as in Appendix 2. It will be observed that the order of $Q_+(\xi)$ at infinity differs from the similar function in the previous case. By using the Stirling expansion formula it may be found that

$$Q_+(\xi) \sim O(\xi^{q-1})$$

and

$$\bar{F}(\xi) \sim O\left(\frac{1}{\xi^{1+q}}\right)$$

where $|\xi| \to \infty$ and $\mathcal{I}m\,\xi < e'$.

This leads to the result

$$f(x) \sim O(x^q) \quad \text{as } x \to 0+.$$

Since $0 < q < \frac{1}{2}$, it will be seen that the initial shape of the boundary lies between that of the straight line $y = h[1 + O(1)]$ and that of the parabola $y = h[1 + O(\sqrt{x})]$. The initial slope of the boundary is infinite and given by $f'(x) = O(x^{-\alpha})$ where $\frac{1}{2} < \alpha = (1-q) < 1$. Thus the initial form of the boundary is that of a continuous curve with an infinity in the gradient at $x = 0$. Although physically impossible it seems reasonable that some discontinuity of this nature could be expected in work of this type based on a linearised theory.

4. The solution for the case $mk = 1$

This case is characterised by the change in position of the zeros and singularities of the expression $\coth k\zeta - m\cot\zeta$ The function has been factorised as shown in equation (56) and it is immediately evident that $Q_+(\zeta)$ of equation (57) has a zero at the origin whereas in the previous two cases, (equations (51) and (53)) $Q_+(\zeta)$ had a pole at the origin. It will be observed that the expression $\tan\zeta - m\tanh k\zeta$ always has a simple zero at the origin; in the case $mk > 1$ there are also two other real zeros at $\pm\sigma_0$; in the case $mk < 1$ there are two imaginary zeros at $\pm i\tau_0$. If now mk be made to tend towards unity in either of these cases the zeros $\pm\sigma_0$ or $\pm i\tau_0$ tend towards zero and in the critical case the expression $[\tan\zeta - m\tanh k\zeta]$ has a triple zero at the origin.

This case, with the change in the condition of $Q_+(\zeta)$ at the origin, provides a solution to the problem which is physically impossible. It will be shown that the solution makes the jet displacement become infinitely large downstream of the jet. Whilst this solution cannot be accepted there is a strong similarity between this solution and that of a problem on waves on the surface of water discussed by J.J. Stoker (25 Pg210) In this latter problem it is shown that in the critical case the infinite solution occurs as in the present problem. However Stoker shows that an infinity occurs, not only in the steady state terms, but also in the transient terms of the general solution of the unsteady problem. Instead of the transient terms decaying with increasing time they are shown, in fact, to increase. It will be shown that in the present problem the solution is correct in so far as the posing of the problem is correct but that the inference is that it is not possible to obtain a valid solution under the assumption of linearised theory. Consider now the solution as obtained under the assumption that the linearised theory is correct.

In the supersonic region the function $\psi(t,y)$ is given by the inverse transform (61) with $\bar{V}_+(\zeta,h)$ given by (47) and $Q_+(\zeta)$ given by (57). Thus,

$$\psi(t,y) = \frac{1}{\sqrt{2\pi}} \int_C \frac{e^{-i\zeta t} \cos(\zeta y/h) \, d\zeta}{\zeta^2 \sin\zeta \, (\coth k\zeta - m\cot\zeta) \, Q_-(\zeta)}$$

in which the integrand has a pole at the origin $\zeta = 0$ of order four and other poles $\zeta = \pm\sigma_r$, $\zeta = \pm r\pi$, $\zeta = -i\tau_r$; r = 1,2,3... The contribution to the function $\psi(t,y)$ from the residue due to the pole at the origin is obtained by calculating the coefficient of ζ^{-1} in the Taylor expansion of the integrand. This expansion may be written,

$$\frac{\left[1 - i\zeta t - \frac{\zeta^2 t^2}{2} + \frac{i\zeta^3 t^3}{6} + O(\zeta^4)\right]\left[1 - \frac{\zeta^2 y^2}{2h^2} + O(\zeta^4)\right]}{\zeta^4\left[1 + \frac{\zeta^2}{6} + O(\zeta^4)\right]\left[\frac{1+k^2}{3k}\right]\left[1 + \frac{3(1-k^2)}{45}\zeta^2 + O(\zeta^4)\right]\left[1 + iC_1\zeta + D_1\zeta^2 + iE_1\zeta^3 + O(\zeta^4)\right]}$$

where C_1, D_1 and E_1 are constants.

It is seen that $\psi(t,y)$ contains terms of the type $(a_1 t^3 + a_2 t y^2 + a_3 t^2 + a_4 y^2 + a_5 t + a_6 y)$ where the a_r are all constants independent of both t and y. It follows that the perturbation potential $\phi(x,y)$ contains terms of the type $(b_1 x^3 + b_2 x y^2 + b_3 y^2 + b_4 x + b_5 y)$ and it is evident that ϕ, $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ all become infinite as $x \to +\infty$. Further, from equation (41) the shape of the jet boundary is given by $y = h[1 + f(x)]$ where

$$h\,f(x) = \int \frac{\partial \phi_+(x,h)}{\partial x}\,dx = c_1 x^2 + c_2 x + c_3 + \text{other terms} \quad x > 0.$$

This relationship shows that the jet leaves the exit with zero gradient and an initial expansion takes place but the boundary continues to diverge. Such a solution is physically impossible but is, nevertheless, the solution of the problem as posed. This leads to the suggestion that the problem is not correctly posed for this critical case, and this is undoubtedly true. The correct solution to the problem will probably not be available until the more difficult non-linear problem has been solved.

Comparison will now be drawn between this problem in gas dynamics and the problem discussed by Stoker[(25)], and already mentioned above, of waves created by a disturbance on the surface of a running stream of finite depth. In this two-dimensional problem the undisturbed stream flows with a velocity U in the region $-h \leq y \leq 0$, $-\infty < x < +\infty$ and

the rigid bottom of the stream is the line $y=-h, -\infty < x < +\infty$.
If a small disturbance of the free surface, measured from the undisturbed position $y = 0$, be assumed, it is possible to adopt the linearised theory and write $\Phi(x,y,t) = Ux + \phi(x,y,t)$ for the velocity potential $\Phi(x,y,t)$. If the motion is also assumed to be steady it is required to find a solution of Laplace's equation $\nabla^2\phi = 0$ subject to the appropriate boundary conditions. It is found that the boundary condition on the free surface leads to an investigation of the roots of the equation $\dfrac{U^2}{gh} = \dfrac{\tanh \zeta h}{\zeta h}$ where g is the acceleration due to gravity. This equation written in the form

$\tanh(\zeta h) - \frac{U^2}{gh}(\zeta h) = 0$, corresponds, in so far as the two problems may be compared, to the equation $\tanh k\zeta - \frac{1}{m}\tan\zeta = 0$ which appears in the problem of the compound jet.

Consider the position of the roots of each equation. For all $\dfrac{U^2}{gh}$, the equation $\tanh(\zeta h) - \frac{U^2}{gh}(\zeta h) = 0$ has a root at the origin and an infinite set of imaginary roots. There are two additional roots the positions of which differ according as $gh > U^2$, $gh = U^2$ or $gh < U^2$. If $gh > U^2$, these roots are real, if $gh = U^2$, these roots are zero and the equation has then a triple root at the origin; if $gh < U^2$, these roots are imaginary and different from any in the above infinite set. In comparison, for all mk, the equation

$\tanh(\zeta k) - \frac{1}{m}\tan\zeta = 0$ has a root at the origin, an infinite set of imaginary roots and an infinite set of real roots. There are two additional roots the positions of which differ according as $mk > 1$, $mk < 1$ or $mk = 1$. If $mk > 1$, these roots are real and different from any of the other real roots; if $mk = 1$, these roots are zero and the equation then has a triple root at the origin; if $mk < 1$, these roots are imaginary and different from any of the roots in the above mentioned infinite set. The important feature of the comparison lies in the fact that both equations have a triple root at the origin in the critical case. Stoker observes that in this case, the solution makes the wave amplitude (and also $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$) tend to infinity with $|x|$. The same observation may be made with regard to the compound jet problem.

In order to make a closer investigation of this critical case Stoker considers the unsteady problem and specifies boundary conditions for time $t > 0$ and initial conditions for the velocity potential, pressure and displacement at t = 0. After applying the Fourier Transform theorem to the field equation (Laplace's equation in his problem) and to the boundary conditions and solving in the usual way he is ~~lead~~ led to the solution for the perturbation potential $\phi(x, y, t)$ in the following form

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x}^{(s)} + \frac{\partial \phi}{\partial x}^{(t)} , \tag{80}$$

where

$$\frac{\partial \phi^{(s)}}{\partial x} = -\frac{U}{\rho\sqrt{2\pi}} \int_C \frac{e^{-i\xi x}\, \bar{p}(\xi)\, \xi^2 \cosh \xi(y+h)\, d\xi}{\cosh \xi h \;\; W(\xi)} \tag{81}$$

$$\frac{\partial \phi^{(t)}}{\partial x} = -\frac{U}{2\rho\sqrt{2\pi}} \int_C \frac{e^{-i\xi x}\, \bar{p}(\xi)\, \xi^2 \cosh \xi(y+h)}{\cosh \xi h \sqrt{g\xi \tanh \xi h}} \left[\frac{e^{-itf_+(\xi)}}{f_+(\xi)} - \frac{e^{-itf_-(\xi)}}{f_-(\xi)} \right] d\xi \tag{82}$$,

and where

$$W(\xi) = \xi^2 U^2 - g\xi \tanh \xi h = (\xi U + \sqrt{g\xi \tanh \xi h})(\xi U - \sqrt{g\xi \tanh \xi h})$$
$$= f_+(\xi) \,.\, f_-(\xi),$$

and $\bar{p}(\xi)$ is the transform of the surface pressure p(x) for $t>0$. The path C is taken from $-\infty$ to $+\infty$ along the real axis in the ξ-plane.

The first term on the right side of (80) is independent of the time and consists of the steady state terms, while the second term is dependent on the time and provides the transient terms of the problem.

An analysis is then taken of the transient solution in each of the three cases. For $U^2 > gh$ or $U^2 < gh$ the transient is found to decay as the time increases and the steady state condition given by (81) does then exist. However, in the critical case $U^2 = gh$ Stoker is able to show that the solution $\frac{\partial \phi^{(t)}}{\partial x}$ is $O(t^{1/3})$ as $t \to \infty$ and the transient does not decay to zero but increases with time. The steady state condition of the problem is thus never attained as a limit of the unsteady problem. Even if the transient terms were, in practice, counterbalanced by dissipative forces the solution of $\frac{\partial \phi^{(s)}}{\partial x}$ given by (81) has a contribution arising from the pole of order 2 at $\xi = 0$ which occurs in the integrand. This contribution gives a term in $\frac{\partial \phi^{(s)}}{\partial x}$ which is linear in x.

The conclusion to be drawn from this discussion is that the linearised theory, which assumes small disturbances, breaks down for this critical case and that the correct solution will be obtained only by an application of the non-linearised equations.

The similarity in the positions of the zeros of the expressions $\tanh(\xi h) - \frac{u^2}{gh}(\xi h)$ for the water wave problem, and $\tanh(\xi k) - \frac{1}{m}\tan \xi$ for the gas jet problem, and, in particular, the fact that both expressions vanish like ξ^3 at $\xi = 0$ in the critical cases suggests that the reasoning adopted by Stoker is also applicable to the compound jet problem. The general, unsteady, compound jet problem is somewhat more difficult than the problem discussed by Stoker. First there are two field equations, instead of one, and second the separation of the terms into time dependent and time independent terms must involve an analysis based on the Wiener-Hopf method of solution. The boundary conditions which replace the equations (9) to (13) are

$$\frac{\partial\phi_+}{\partial y}(x, h+0, t) = \frac{\partial\phi_+}{\partial y}(x, h-0, t) = v_+(x, h, t) \quad , \qquad (9a)$$

$$\frac{\partial\phi_-}{\partial y}(x, h+0, t) = \frac{\partial\phi_-}{\partial y}(x, h-0, t) = 0 \quad , \qquad (10a)$$

$$\frac{\partial\phi}{\partial y}(x, 0, t) = 0 \quad , \qquad (11a)$$

$$\frac{\partial\phi}{\partial y}(x, H-0, t) = 0 \quad , \qquad (12a)$$

$$\left(1+\frac{1}{W_2}\right)\frac{\partial\phi_+(x, h+0, t)}{\partial x} - \ell^2\left(1+\frac{1}{W_1}\right)\frac{\partial\phi_+(x, h-0, t)}{\partial x} = -\varepsilon \qquad (13a)$$

The field equations which replace (14) and (15) are

$$\frac{\partial^2\phi}{\partial y^2} - \beta_1^2\frac{\partial^2\phi}{\partial x^2} = \frac{1}{a_1^2}\left[\frac{\partial^2\phi}{\partial t^2} + 2W_1\frac{\partial^2\phi}{\partial x\,\partial t}\right] \qquad (14a)$$

$$0 \leq y \leq h-0, \quad -\infty < x < +\infty,$$

and

$$\frac{\partial^2\phi}{\partial y^2} + \beta_2^2\frac{\partial^2\phi}{\partial x^2} = \frac{1}{a_2^2}\left[\frac{\partial^2\phi}{\partial t^2} + 2W_2\frac{\partial^2\phi}{\partial x\,\partial t}\right] \qquad (15a)$$

$$h+0, \leq y \leq H-0, \quad -\infty < x < +\infty.$$

If the assumption be made that the jet and outer stream have undisturbed flows at $t = 0$, the initial conditions are

$$\phi(x, y, 0) = \frac{\partial\phi(x, y, 0)}{\partial t} = 0.$$

It is then necessary to produce an integral solution for ϕ (or $\frac{\partial\phi}{\partial x}$) in which the transient and steady state terms are separated. If a Fourier-Laplace transfom be applied to the equations (9a) to (15a) an equation analogous to (28) will be obtained. This takes the form

$$\overset{\approx}{V}_+\left[\frac{W_2\alpha+p}{W_2}\,\frac{a_2}{q_2}\coth\left(\frac{q_2 L}{a_2}\right)+l^2\frac{W_1\alpha+p}{W_1}\,\frac{a_1}{q_1}\coth\left(\frac{q_1 h}{a_1}\right)\right]$$

$$=\frac{\varepsilon}{\alpha p}-\left[\frac{W_2\alpha+p}{W_2}\,\overset{\approx}{\phi}_-(\alpha,h+0,p)\right], \tag{83}$$

where
$$q_1^2=(W_1\alpha+p)^2-a_1^2\alpha^2$$
$$q_2^2=(W_2\alpha+p)^2-a_2^2\alpha^2 \quad ,$$

and α is the complex variable in the Fourier Transform and p the complex variable in the Laplace transform, so that

$$\bar{\phi}(\alpha,y,t)=\bar{\phi}_++\bar{\phi}_-=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{i\alpha x}\phi_+\,dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{i\alpha x}\phi_-\,dx$$

$$\overset{\approx}{\phi}(\alpha,y,p)=\int_0^\infty e^{-pt}\,\bar{\phi}(\alpha,y,t)\,dt\,,$$

and $\overset{\approx}{V}_+(\alpha,h,p)$ is the Fourier-Laplace transform of $\frac{\partial\phi_+}{\partial y}$. It has not been found possible to furnish a solution to this general problem in which the transient and steady state terms are separated but it seems reasonably certain that if such a solution were found the transient terms would be seen to increase with time in the critical case and, as in Stoker's problem, the steady state would never be attained. The further discussion of the critical case solution must therefore await the solution of the unsteady problem.

5. Conclusions

In this chapter the solutions of the problem of a supersonic jet stream embedded in a subsonic finite stream have been obtained and discussed. The solutions vary according as whether the value of the relationship $\frac{\gamma_1 M_1^2 \beta_2^2 L}{\gamma_2 M_2^2 \beta_1^2 h}$ $(= mk)$ between the gas constants, the Mach numbers and the widths of the streams, is greater than, equal to or less than unity. The general solutions are given in each case. It is of interest to note that, in practice, the solution for $mk > 1$ is of most importance. Remember that $M_1^2 > 1$; $M_2^2 < 1$; $L > h$; and that $\beta_1^2 = M_1^2 - 1$ and $\beta_2^2 = 1 - M_2^2$. It follows that mk exceeds unity if $M_2 < \frac{1}{\sqrt{2}}$. Due to the assumptions of the linearised theory, for $M_2 > \frac{1}{\sqrt{2}}$, the solutions may be expected to be inaccurate since the problem is one in which the stream velocity is approaching the sonic speed. If, in fact, the problem be further restricted so that the transonic speeds be avoided altogether, that is the velocities are such that $M_1 > \sqrt{2}$ and $M_2 < \frac{1}{\sqrt{2}}$ then it is clear that $mk > \frac{L}{h} > 1$. The inequality $M_2 < \frac{1}{\sqrt{2}}$ must be subject to the condition that $\varepsilon \left(= \frac{P_1 - P_2}{\rho_2 W_2^2} \right)$ is kept small. The importance of the case $mk > 1$ is further emphasised since it includes the problem in which the outer stream is of infinite width $(k \to \infty)$. The general solution for this case has been given, in a series form, by equations (65) to

(68) and it has been shown that the results given by Pai are just the asymptotic form of this solution far downstream of the jet exit. The analysis also shows that far downstream, in the general case, the jet boundary varies about a mean displacement $\frac{\varepsilon h \beta_1 \beta_2 k}{mk-1}$ measured from the undisturbed position of the boundary. In the overpressure case, when $\varepsilon > 0$, this displacement is positive or negative according as mk is greater than or less than unity. Some remarks are made at the end of Chapter IV concerning this dependency on mk.

In the discussion for the case mk = 1 it is found that the displacements and velocities become large at large distance downstream. This situation is also seen to arise in a problem on water waves discussed by Stoker and the similarity between the solutions of the two problems leads to the same conclusions; that the assumption of linearised theory breaks down for this case. It would seem that much information could be obtained, in each of the three cases, if the general unsteady state problem could be solved in such a way that the transient and steady state terms could be examined separately.

CHAPTER II

The flow of a supersonic two-dimensional gas jet in a uniform supersonic flow

This problem, with the outer stream of infinite width, was originally investigated by Pai[13] who obtained a solution by the method of characteristics. Since both flows are supersonic the streams are undisturbed upstream of the leading characteristics from the jet exit. From the general solution, in the form $\phi_1 = \sum f_n(x-\beta_1 y) + \sum g_n(x+\beta_1 y)$ and $\phi_2 = \sum F_n(x-\beta_2 |y|)$ in which f_n, g_n and F_n are arbitrary, Pai analysed a simple case in which the perturbation velocity at the jet exit was AU_1 where U_1 was the undisturbed velocity in the jet and A a constant. He observed that there was no periodic, or almost periodic, structure in the jet and examined the initial and reflected disturbances.

Later, Pack[15] found the general solution of this problem by using the (one-sided) Laplace transform, and examined the wave structure in the jet and the fluctuations in the jet boundary by expanding the inverse transforms in powers of exponential functions with negative exponents[22 Pg 192]. The behaviour of the jet was found to depend on the value of the relationship $\dfrac{\gamma_1 M_1^2 \beta_2}{\gamma_2 M_2^2 \beta_1}$ between the gas constants, and the velocities of the undisturbed stream and jet. Pack showed that there was a singular solution when this relationship was unity and that the jet boundary expanded for a certain distance and thereafter the width of the jet remained constant. He confirmed the findings of Pai.

The work of both Pack and Pai was further generalised by Ehlers and Strand (16) who discussed the case of a supersonic jet inclined at a small angle to the main supersonic stream.

In the present chapter the problem of a supersonic jet which flows into a supersonic stream of finite width will be examined by using the technique already developed in the previous chapter. It is not strictly necessary to use the Wiener-Hopf method to extract the solution since the perturbation effects take place downstream of the jet exit and the transform integrals are taken from $x = 0$ to $x = +\infty$. Nevertheless, it is of interest to reformulate the problem of Chapter 1 so that the outer stream becomes supersonic. It will be found that the problem leads to an equation corresponding to equation (36) but that no factorisation is necessary and that the solution may be written down almost at once. It will be shown (see, for example, Figure 2) that one effect of introducing the outer wall is to create, for certain values of the parameters of the problem, a periodicity in the movement of the jet boundary.

In the notation of the previous chapter the boundary conditions are unchanged and are given by (3), (4), (5) and (6); the field equation for the perturbation potential in the jet is given by (7) and the field equation for the perturbation potential in the outer stream is given by equation (8) but it must be noted that $M_2 > 1$. Let equation (8) be written

$$\frac{\partial^2 \phi_2}{\partial y^2} - (\beta_2')^2 \frac{\partial^2 \phi_2}{\partial x^2} = 0$$

where $(\beta_2')^2 = M_2^2 - 1$.

The results which follow equation (8) of Chapter 1 will now hold true in the present problem if β_2 be replaced by $i\beta_2'$ Proceed as in Chapter 1 and it may be shown that equations (28) to (33) are replaced by

$$\frac{1}{i\beta_2'} \bar{\vartheta}_+(\alpha, h)\, K'(\alpha) = -\frac{\varepsilon}{\alpha\sqrt{2\pi}} + G_-'(\alpha, h) \qquad ,$$

$$K'(\alpha) = \coth(i\alpha\beta_2' L) - i m' \cot(\alpha\beta_1 h) \qquad ,$$

$$m' = \frac{\gamma_1 M_1^2 \beta_2'}{\gamma_2 M_2^2 \beta_1} \qquad ,$$

$$L = H - h \qquad ,$$

$$\bar{\vartheta}_+(\alpha, h) = \frac{d\bar{\Phi}_+(\alpha, h)}{dy} \qquad ,$$

and

$$G_-'(\alpha, h) = \alpha\left[\bar{\Phi}_-(\alpha, h+0) - \ell^2 \bar{\Phi}_-(\alpha, h-0)\right].$$

Due to the nature of supersonic flow it is evident that $G_-'(\alpha, h)$ is identically zero. After noting that $K'(\alpha)$ may be written $-i\left[\cot(\alpha\beta_2' L) + m' \cot(\alpha\beta_1 h)\right]$, the primes may be dropped and the equation becomes

$$\frac{\bar{\vartheta}_+(\alpha, h)\; K(\alpha)}{\beta_2} = \frac{\varepsilon}{\alpha\sqrt{2\pi}} \qquad , \qquad (84)$$

where

$$K(\alpha) = \cot(\alpha\beta_2 L) + m\cot(\alpha\beta_1 h), \qquad (85)$$

$$m = \frac{\gamma_1 M_1^2 \beta_2}{\gamma_2 M_2^2 \beta_1} \tag{86}$$

$$L = H - h \tag{87}$$

$$\bar{v}_+(\alpha, h) = \frac{d\,\bar{\Phi}_+(\alpha, h)}{dy} \tag{88}$$

and $\beta_1^2 = M_1^2 - 1$, $\beta_2^2 = M_2^2 - 1$.

The transformations (43)

$$\left.\begin{array}{ll} \xi = \alpha \beta_1 h , & k = \beta_2 L / \beta_1 h \\ t = x / \beta_1 h , & v_+(x,y) = \varepsilon \beta_2 V_+(t,y) / \sqrt{2\pi} \end{array}\right\} \tag{43}$$

are used to show that
$$\bar{V}_+(\xi, h)\; Q(\xi) = 1/\xi$$

where
$$Q(\xi) = \cot k\xi + m \cot \xi \tag{89}$$

In this result $\bar{V}_+(\xi, h)$ is regular in the upper half plane $\mathcal{I}m\,\xi > 0$, the right side , $1/\xi$, is regular in the same region and it remains to discuss the function $Q(\xi)$. Now, it may be shown (26 Pg324) that $[\cot k\xi + m \cot \xi]$ has all of its poles and zeros real and simple. It follows that $Q(\xi)$ is regular and free of zeros in the region $\mathcal{I}m\,\xi > 0$. Thus both sides of (89) are analytic and non-zero in this upper half plane. The equation

$$\bar{V}_+(\xi, h) = \frac{1}{\xi\, Q(\xi)} \tag{90}$$

then takes the place of (47) of the previous Chapter.

The jet displacement is defined by equation (40). With the transformation $f(x) = i\varepsilon\beta_1\beta_2 F(t)/\sqrt{2\pi}$ of (44), the equations (42) and (89) lead to

$$\bar{F}(\xi) = \frac{1}{\xi^2 Q(\xi)} \tag{91}$$

The intermediate variable $\psi(t,y)$ given by (44) may be introduced, and equation (90) leads to the following inverse transforms (compare with (61) and (62)).

$$\psi(t,y) = -\frac{1}{\sqrt{2\pi}}\int_C \frac{e^{-i\xi t}\cos(\xi y/h)\;\bar{V}_+(\xi,h)\;d\xi}{\xi\sin\xi}$$

$$= -\frac{1}{\sqrt{2\pi}}\int_C \frac{e^{-i\xi t}\cos(\xi y/h)\;d\xi}{\xi^2\sin\xi.(\cot k\xi + m\cot\xi)}, \tag{92}$$

$$0 \leq y \leq h-0,$$

and

$$\psi(t,y) = -\frac{1}{\sqrt{2\pi}}\frac{\beta_1}{\beta_2}\int_C \frac{e^{-i\xi t}\cos[k\xi(H-y)/L]\;\bar{V}_+(\xi,h)\,d\xi}{\xi.\sin k\xi}$$

$$= -\frac{1}{\sqrt{2\pi}}\frac{\beta_1}{\beta_2}\int_C \frac{e^{-i\xi t}\cos[k\xi(H-y)/L]\;d\xi}{\xi^2.\sin k\xi.[\cot k\xi + m.\cot\xi]} \tag{93}$$

$$h+0 \leq y \leq H-0,$$

where C is a path drawn from $-\infty + i\gamma$ to $+\infty + i\gamma$ and γ is a small positive constant.

The fluctuations of the jet boundary are studied by examination of the function $F(t)$ given, from (91), by

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_C \frac{e^{-i\xi t}\, d\xi}{\xi^2 (\cot k\xi + m \cot \xi)} \tag{94}$$

Once again the path of integration may be closed by an infinite semicircle drawn in either the upper or lower half plane. The semicircle is drawn so as to pass between the poles of the integrands on the real axis. The contribution to the integral from the infinite semicircle may be shown to vanish and $\psi(t, y)$ is then obtained from the residues at the poles within the contour. Suppose the path to be closed by the semicircle drawn in the upper half plane. There are no singularities inside the contour and hence $\psi(t, y)$ is zero in the region $t + y/h - 1 < 0$, $0 \leq y \leq h-0$. Likewise, in (93), if the path be closed in the upper half plane it is seen that $\psi(t, y)$ is zero in the region $t + k(H-y)/L - k < 0$, $h+0 \leq y \leq H-0$. In terms of the original variables the perturbation potential $\phi_1(x,y)$, in the jet, is zero upstream of the leading characteristic (of equation $y - h + x/\beta_1 = 0$) from the edge of the jet exit. If the transformation (43) be used, it may be found that the perturbation potential $\phi_2(x,y)$, in the stream, is zero upstream of the leading characteristic (of equation $h - y + x/\beta_2 = 0$) from the edge of the jet exit.

It will presently be shown that a solution may be obtained in terms of the other characteristics of the flows (see equations (101) and (102)).

If the path of integration in (92) be closed with an infinite semicircle taken in the lower half plane then the contour integral is taken round a region in which the integrand has poles at the zeros of $\zeta^2 \sin\zeta\,(\cot k\zeta + m\cot\zeta)$. For all k, rational or irrational, the integrand has a double pole at the origin and other poles at $\zeta = \pm\sigma_r$, r = 1,2,3... where the σ_r are the positive roots of $\tan\zeta + m\tan k\zeta = 0$.

Write

$$\frac{-i\,\psi(t,y)}{\sqrt{2\pi}} = R_1 + R_2$$

where R_1 is the residue at the pole $\zeta = 0$

and R_2 is the residue at the poles $\zeta = \pm\sigma_r$.

The residue at $\zeta = 0$ is the coefficient in the Taylor expansion of the integrand at the origin. This expansion is

$$\frac{\left[1 - i\zeta t + O(\zeta^2)\right]\left[1 + O(\zeta^2)\right]}{\zeta^2\,(1/k + m)\left[1 + O(\zeta^2)\right]}$$

The residue at $\zeta = 0$ is thus $-\dfrac{itk}{mk+1}$.

The residue at $\zeta = \pm\sigma_r$ is given by

$$\frac{\cos(\sigma_r y/k)\; 2i\sin\sigma_r t}{\sigma_r^2 . \sin\sigma_r . \left[k\,\mathrm{cosec}^2 k\sigma_r + m\,\mathrm{cosec}^2\sigma_r\right]}$$

Thus $$\frac{\psi(t,y)}{\sqrt{2\pi}} = \frac{kt}{mk+1} - 2\sum_{1}^{\infty} \frac{\cos(\sigma_r y/h) \quad \sin \sigma_r t}{\sigma_r^2 \quad \sin\sigma_r \left(k \operatorname{cosec}^2 k\sigma_r + m \operatorname{cosec}^2 \sigma_r\right)}$$

and finally

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = \frac{xk}{\beta_1 h(mk+1)} - 2\sum_{1}^{\infty} \frac{\cos(\sigma_r y/h) \sin(\sigma_r x/\beta_1 h)}{\sigma_r^2 \sin\sigma_r. \left(k \operatorname{cosec}^2 k\sigma_r + m \operatorname{cosec}^2\sigma_r\right)}$$

$$x/\beta_1 + y - h > 0$$
$$0 \leq y \leq h-0,$$

and $$\frac{\phi(x,y)}{\varepsilon h \beta_2} = 0, \quad x/\beta_1 + y - h < 0$$
$$0 \leq y \leq h-0.$$

In like fashion the solution can be obtained within the region of the outer stream. The form of the equations (92) and (93) are similar and it follows that

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} = \frac{x}{\beta_1 h(mk+1)} - 2\sum_{1}^{\infty} \frac{\cos[k\sigma_r(H-y)/L] \ \sin(\sigma_r x/\beta_1 h)}{\sigma_r^2 \ \sin k\sigma_r \ \left(k \operatorname{cosec}^2 k\sigma_r + m \operatorname{cosec}^2\sigma_r\right)},$$

$$h - y + x/\beta_2 > 0$$
$$h+0 \leq y \leq H-0,$$

and

$$\frac{\phi(x,y)}{\varepsilon \ell \beta_1} = 0, \quad h - y + x/\beta_2 < 0,$$
$$h+0 \leq y \leq H-0.$$

The fluctuations in the jet boundary may be examined by an investigation of the function F(t) given by

$$\bar{F}(\xi) = \frac{1}{\xi^2 Q(\xi)} \tag{91}$$

Now, $$F(t) = \frac{1}{\sqrt{2\pi}} \int_C \frac{e^{-i\zeta t} \, d\zeta}{\zeta^2 \left(\cot k\zeta + m \cot \zeta \right)}$$

and the integrand has poles at $\zeta = 0$ and $\zeta = \pm \sigma_r$.
Thus

$$\frac{i\,F(t)}{\sqrt{2\pi}} = \frac{1}{(1/k + m)} - 2\sum_1^\infty \frac{\cos \sigma_r t}{\sigma_r^2 \left(k \operatorname{cosec}^2 k\sigma_r + m \operatorname{cosec}^2 \sigma_r \right)}$$

and hence, from (44),

$$\frac{f(x)}{\varepsilon \beta_1 \beta_2} = \frac{k}{mk+1} - 2\sum_1^\infty \frac{\cos (\sigma_r x / \beta_1 h)}{\sigma_r^2 \left(k \operatorname{cosec}^2 k\sigma_r + m \operatorname{cosec}^2 \sigma_r \right)} \qquad (95)$$

If use be made of the result $\cot k\sigma_r = -m \cot \sigma_r$ the denominator inside the summation may be simplified, thus

$$\frac{f(x)}{\varepsilon \beta_1 \beta_2} = \frac{k}{mk+1} - 2\sum_1^\infty \frac{\cos (\sigma_r x / \beta_1 h)}{\sigma_r^2 \left[(k+m) + (k + 1/m) \cot^2 k\sigma_r \right]}$$

The jet boundary therefore expands (for $\varepsilon > 0$) to a mean width given by $y = h \left[1 + \frac{\varepsilon \beta_1 \beta_2 k}{mk+1} \right]$ with a variation from this mean. The variation is maintained as $x \to +\infty$ and hence the ultimate stream width has a mean value given by $f(x) = \frac{\varepsilon \beta_1 \beta_2 k}{mk+1}$. If the width of the outer stream be made infinite then k is infinite, and the ultimate mean width is given by $f(x) = \frac{\varepsilon \beta_1 \beta_2}{m}$ which is just the value found by

Pack[15]. The extra terms which remain in the present problem are due entirely to the interference arising from the waves reflected from the outer wall.

The condition of the jet boundary, for any x, can best be obtained by an observation of the wave formation in the boundary. This is achieved by expanding $\bar{F}(\xi)$ in powers of exponential functions. To this end, and also to draw comparison more easily with the problem discussed by Pack, the problem will be framed in terms of the Laplace transform instead of the Fourier transform.

Write the Laplace transform of $\phi(x,y)$ as $\bar{\phi}(\alpha, y)$ such that

$$\bar{\phi}(\alpha, y) = \int_0^\infty e^{-\alpha x} \phi(x, y)\, dx .$$

Equations (84) and (85) will be replaced by

$$\frac{\bar{\vartheta}_+(\alpha, h)\; K(\alpha)}{\beta_2} = \frac{\varepsilon}{\alpha}$$

with

$$K(\alpha) = \coth \alpha\beta_2 L + m \coth \alpha\beta_1 h .$$

The transformations

$$\xi = \beta_1 h \alpha \;, \quad k = \beta L / h \;, \quad \beta = \beta_2/\beta_1$$

$$t = x/\beta_1 h \;, \quad \vartheta_+(x, y) = \varepsilon \beta_2 V_+(t, y),$$

lead to

$$\bar{V}_+(\xi, h)\; Q(\xi) = 1/\xi$$

with

$$Q(\xi) = \coth k\xi + m \coth \xi , \tag{96}$$

where $\bar{V}_+(\xi, h)$ and $Q(\xi)$ are analytic and non-zero in the region $Re\, \xi > 0$.

Also, put $f(x) = \varepsilon \beta_1 \beta_2 F(t)$ and then

$$\bar{F}(\xi) = \frac{1}{\xi^2 Q(\xi)}$$

$$= \frac{1}{\xi^2 (\coth k\xi + m \coth \xi)} \tag{97}$$

This equation may be compared with the corresponding one obtained by Pack for the case of an infinite outer stream, namely

$$\bar{F}(\xi) = \frac{1}{\xi^2 (1 + m \coth \xi)}$$

Now expand the function $\bar{F}(\xi)$ of (97) in ascending powers of $e^{-2\xi}$ and $e^{-2k\xi}$ and transform term by term.

Write

$$\bar{F}(\xi) = \frac{1}{\xi^2} \frac{1}{\left(\frac{1+b}{1-b} + m \frac{1+a}{1-a} \right)}$$

where $a = e^{-2\xi}$ and $b = a^k = e^{-2k\xi}$. Then, with $\mu = \frac{1-m}{1+m}$,

$$\bar{F}(\xi) = \frac{1}{(1+m)\xi^2} \left[1 - (a+b) + ab \right] \left[1 + (ab + \mu\{a-b\}) + (ab+\mu\{a-b\})^2 + --- \right]$$

Consider only the first few terms of the series, then $\bar{F}(\xi)$

may be written

$$(1+m)\,\bar{F}(\zeta) = \frac{1}{\zeta^2}\Big[1 - \{(1-\mu)e^{-2\zeta} + (1+\mu)e^{-2k\zeta}\} - \{\mu(1-\mu)e^{-4\zeta} - 2(1-\mu^2)e^{-2(1+k)\zeta} - \mu(1+\mu)e^{-4k\zeta}\} + \cdots\Big].$$

On transforming term by term

$$\begin{aligned}(1+m)\,F(t) = \Big[\, &\{t\} - (1-\mu)\{t-2\} - (1+\mu)\{t-2k\} \\ &-\mu(1-\mu)\{t-4\} + 2(1-\mu^2)\{t-2k-2\} \\ &+\mu(1+\mu)\{t-4k\} + \cdots \Big]\end{aligned} \qquad (98)$$

where $\{t-T\}$ stands for $(t-T)H(t-T)$ and

$$\begin{aligned} H(t-T) &= 1, \quad t \geqslant T, \\ &= 0, \quad t < T. \end{aligned}$$

At the jet exit, t is zero and the slope of the boundary $F'(t)$ is $1/(1+m)$, a result in agreement with that given by Pack. The fluctuations of the boundary are complicated by the arrival of both incident and reflected waves inside the jet and by reflected waves from outside the jet. For instance, if $k = 3/4$ the slope of the jet for $0 < t < 3/2$ is $1/(1+m)$, for $3/2 < t < 2$ is $-\mu/(1+m)$, for $2 < t < 3$ is $-1/(1+m)$, for $3 < t < \frac{7}{2}$ is $-(1-\mu-\mu^2)/(1+m)$ and so on. The case $k = 1$ is of particular interest. The function $\bar{F}(\zeta)$ of (97) takes the form

$$\bar{F}(\xi) = \frac{\tanh\xi}{(1+m)\xi^2} = \frac{1}{(1+m)\xi^2}\left[1 - 2e^{-2\xi}\left(1 - e^{-2\xi} + e^{-4\xi} - e^{-6\xi} + \cdots\right)\right]$$

The solution $F(t)$ is given by

$$F(t) = \frac{1}{1+m}\left[\{t\} - 2\{t-2\} + 2\{t-4\} - 2\{t-6\} + \cdots\right] \qquad (99)$$

The graph of $F(t)$ against t is a well-known triangular waveform (27 Pg 55). In this case the boundary of the jet oscillates with a slope given by $F'(t) = \pm 1/(1 + m)$ and the waves reflected from the outer walls and the incident and reflected waves within the jet meet at the same points along the jet boundary (see Figure 2(a)). The oscillations are then maintained downstream and the jet boundary takes a mean width given by $F(x) = \varepsilon\beta_1\beta_2/(1+m)$. The result given by equation (99) is in agreement with the representation by equation (95). For, if k be put equal to unity and it be noted that $\sigma_r = (2r+1)\pi/2$ (the zeros of $\cot\xi = 0$) then

$$\frac{F(x)}{\varepsilon\beta_1\beta_2} = \frac{1}{(1+m)} - \frac{8}{\pi^2(1+m)}\sum_{1}^{\infty}\frac{\cos\left[(2n+1)\pi x/2\beta_1 h\right]}{(2n+1)^2}$$

and this is just the Fourier series for the triangular waveform specified by (99).

If the outer stream width be made infinitely large, that is if H be made to approach infinity, then k approaches infinity and the point of arrival of the first reflected wave from the outer stream wall is infinitely far downstream. In this case the solution given by (98) reduces to that given by Pack. In his solution Pack observes that there is a singular

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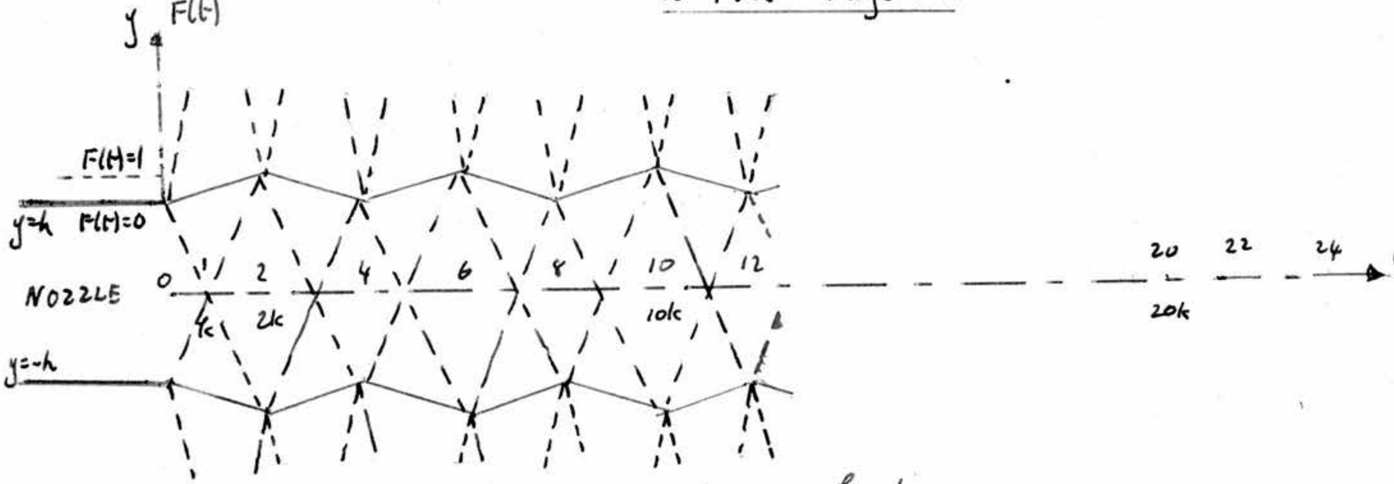

Figure 2(a). $k = 1$.

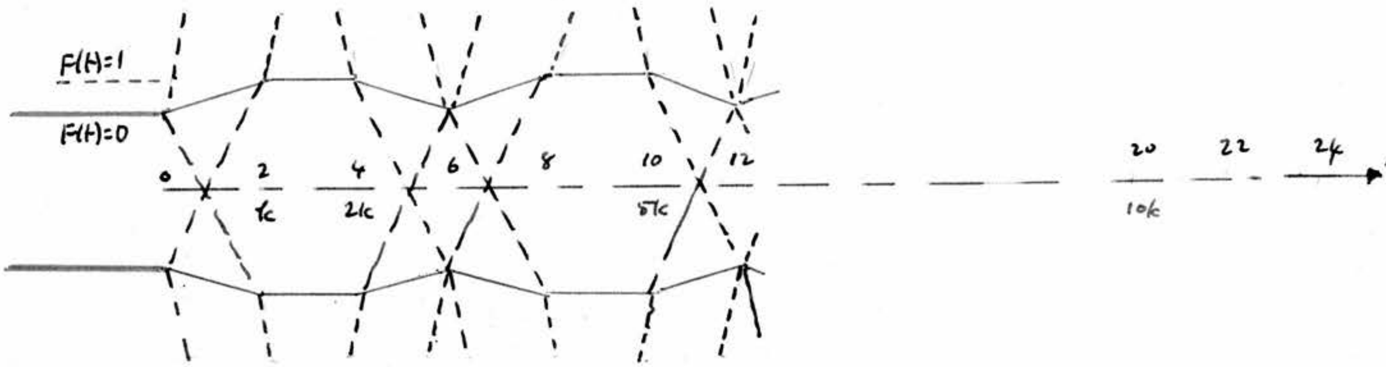

Figure 2(b) $k = 2$, $m = 1$.

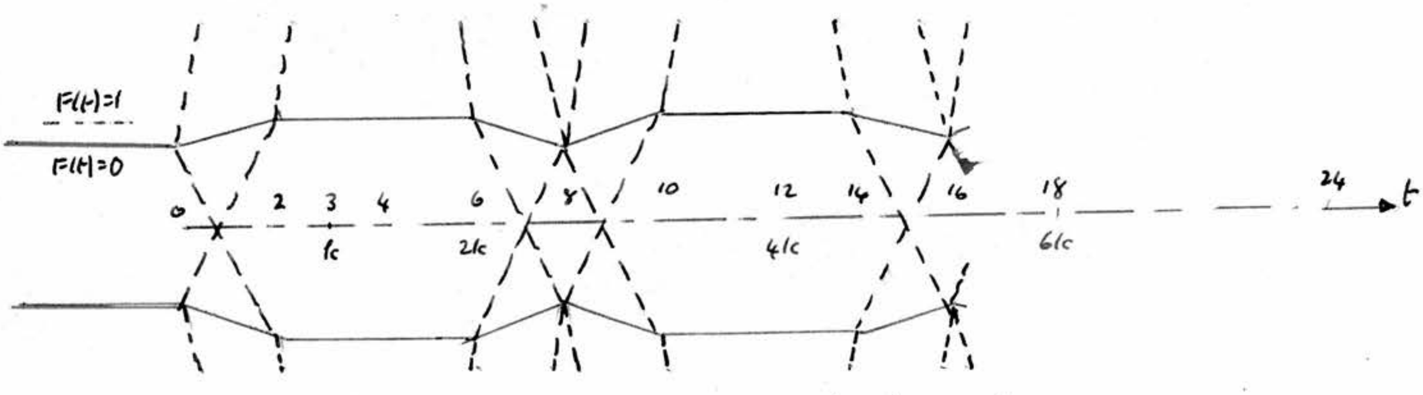

Figure 2(c) $k = 3$, $m = 1$.

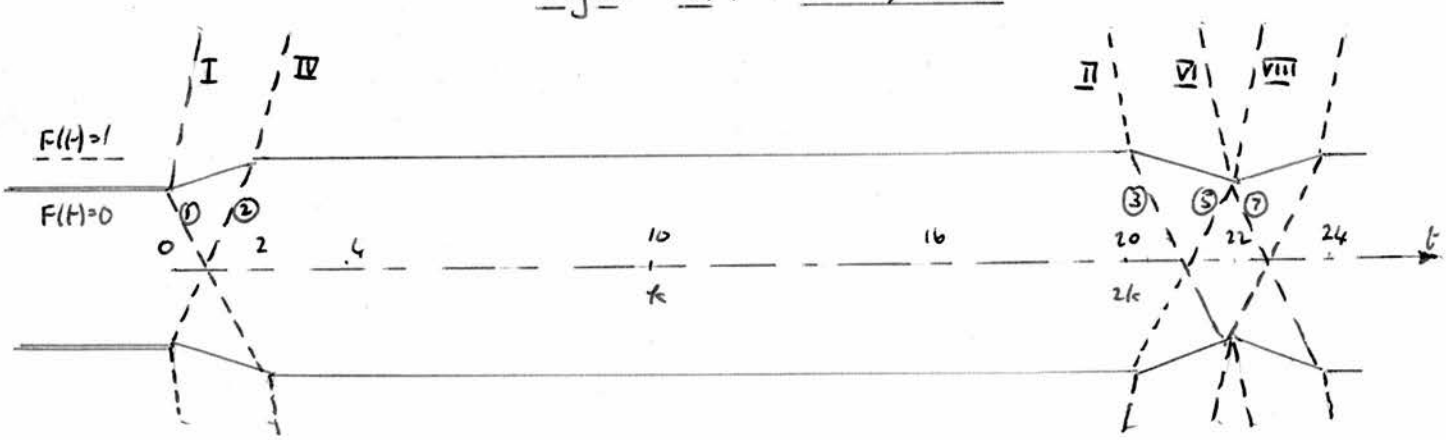

Figure 2(d) $k = 10$, $m = 1$.

case when $\beta_1 \beta_2 = 1$ (that is, when $m = 1$) and the jet boundary then expands to a width given by $F(t) = 1$, or $f(x) = \varepsilon$, and thereafter remains at this constant width. It is of interest to see how this case arises as the transition is made from a finite to an infinite outer stream. Consider the expansion of $\bar{F}(\xi)$ with $\mu = 0, (m=1)$,

$$\bar{F}(\xi) = \frac{1}{2\xi^2}\left[1-(a+b)+ab\right]\left[1+ab+a^2b^2+a^3b^3+\cdots\right]$$

where $a = e^{-2\xi}$ and $b = a^k$, and take k a positive integer. The case k = 1 has already been discussed (figure 2(a)). The boundary expands to F(2) = 1 and then oscillates as shown between F(t) = 1 and F(t) = 0 .

Let k = 2 , $\bar{F}(\xi)$ now takes the form, with $b = a^2$,

$$\bar{F}(\xi) = \frac{1}{2\xi^2}\left[1-a-a^2+2a^3-a^4-a^5+2a^6-\cdots\right]$$

and hence,

$$F(t) = \frac{1}{2}\left[\{t\} - \{t-2\} - \{t-4\} + 2\{t-6\}\cdots\right]$$

This waveform is shown in Figure 2(b). It will be observed that the boundary expands to F(2) = 1, remains constant in the interval $2 \leq t \leq 4$ and then contracts to F(6) = 0. This waveform is repeated as t increases. The mean value of F(t) is 2/3 and this agrees with the value k/(mk + 1) of (95), with k = 2 and m = 1.

Let k = 3, then

$$\bar{F}(\xi) = \frac{1}{2\xi^2}\left[1-a-a^3+2a^4-a^5-a^7+2a^8-\cdots\right]$$

and hence,

$$F(t) = \frac{1}{2}\left[\{t\} - \{t-2\} - \{t-6\} + 2\{t-8\} - \cdots \right]$$

The waveform in this case is shown in Figure 2(c). The jet expands to $F(2) = 1$, remains constant over the interval $2 \leq t \leq 6$ and then contracts to $F(8) = 0$. This waveform is repeated as t increases. The mean displacement is given by $F(t) = 3/4$.

Let $k = 10$, then

$$\bar{F}(\xi) = \frac{1}{2\xi^2}\left[1 - a - a^{10} + 2a^{11} - a^{12} - a^{21} + 2a^{22} - \cdots \right]$$

and hence,

$$F(t) = \frac{1}{2}\left[\{t\} - \{t-2\} - \{t-20\} + 2\{t-22\} - \cdots \right]$$

The waveform in this case is shown in Figure 2(d). The jet now expands to $F(2) = 1$, remains constant over the interval $2 \leq t \leq 20$ and then contracts to $F(22) = 0$. The waveform is repeated as t increases, the mean displacement being $F(t) = 10/11$.

As k increases the disturbance due to the arrival of the reflected wave from the outer stream wall moves further downstream and disappears as k becomes infinitely large. The jet boundary expands to $F(2) = 1$ and then remains constant. It may be observed that for $m = 1$ the oscillation in the jet boundary is periodic, with a period, in t, of $2(k + 1)$, when k is integral. This is also true when k is rational but not when k is irrational. In this latter case the waveform is, however, of a similar shape but is non-periodic.

Further investigation may be made into the nature of the reflected and transmitted waves by expanding the inverse transforms of the perturbation potentials $\psi(t, y)$, in each of the two regions, in powers of exponential functions. Suppose, as in the above analysis, the Laplace transform be used then the perturbation potentials are given by

$$\bar{\psi}(\xi, y) = \frac{\cosh(\xi y/h) \quad \bar{V}_+(\xi, h)}{\xi \quad \sinh \xi}$$

$$= \frac{\cosh(\xi y/h)}{\xi^2 . \sinh \xi . \quad (\coth k\xi \ + m \coth \xi)} \tag{99}$$

$$0 \leq y \leq h-0, \quad -\infty < x < +\infty,$$

and

$$\bar{\psi}(\xi, y) = \frac{\cosh[k\xi(H-y)/L]}{\beta \xi^2 \quad \sinh k\xi \quad (\coth k\xi + m \coth \xi)} \tag{100}$$

$$h+0 \leq y \leq H-0, \quad -\infty < x < +\infty,$$

where $\beta = \beta_2/\beta_1$.

Consider the expansion of (99) in the form

$$\bar{\psi}(\xi, y) = \frac{1}{\xi^2} \; e^{-\xi(1-y/h)} \left[1 + e^{-2\xi y/h}\right] \left[1 + e^{-2\xi} + e^{-4\xi} + \cdots\right] \left[g(\xi)\right],$$

where

$$
\begin{aligned}
g(\xi) &= [\coth k\xi + m\coth\xi]^{-1} \\
&= \frac{1}{(1+m)}\Big[\,1 - (1-\mu)\left(e^{-2\xi} + \mu e^{-4\xi} + \mu^2 e^{-6\xi} + \cdots\right) \\
&\qquad - (1+\mu)\left(e^{-2k\xi} - \mu e^{-4k\xi} + \mu^2 e^{-6k\xi} + \cdots\right) \\
&\qquad + 2(1-\mu^2)\, e^{-2(1+k)\xi} + \cdots \\
&\qquad - (1-\mu^2)(1-3\mu)\, e^{-2(2+k)\xi} + \cdots \\
&\qquad - (1-\mu^2)(1+3\mu)\, e^{-2(1+2k)\xi} + \cdots \Big] \\
&\qquad\qquad 0 \leq y \leq h-0 \quad .
\end{aligned}
$$

This may be written

$$
\begin{aligned}
(1+m)\,\bar{\psi}(\xi, y) &= \frac{1}{\xi^2}\, e^{-\xi(1-y/h)}\Big[\,1 + e^{-2\xi y/h} - (1+\mu)\, e^{-2k\xi} + \\
&\quad + \mu e^{-2\xi} - (1+\mu)\, e^{-2\xi(k+y/h)} + \mu(1+\mu)\, e^{-4k\xi} \\
&\quad + (1+\mu)(1-2\mu)\, e^{-2(1+k)\xi} + \mu^2 e^{-2(2+y/h)\xi} + \cdots \Big]
\end{aligned}
$$

and this gives, on interpreting term by term,

$$
\begin{aligned}
(1+m)\,\psi(t, y) &= \Big[\left\{t + \tfrac{y}{h} - 1\right\}_1 + \left\{t - \tfrac{y}{h} - 1\right\}_2 - (1+\mu)\left\{t + \tfrac{y}{h} - 1 - 2k\right\}_3 \\
&\quad + \mu\left\{t + \tfrac{y}{h} - 3\right\}_4 - (1+\mu)\left\{t - \tfrac{y}{h} - 1 - 2k\right\}_5 \\
&\quad + \mu(1+\mu)\left\{t + \tfrac{y}{h} - 1 - 4k\right\}_6 + (1+\mu)(1-2\mu)\left\{t + \tfrac{y}{h} - 3 - 2k\right\}_7 \\
&\quad + \mu^2\left\{t + \tfrac{y}{h} - 4\right\}_8 + \cdots \Big] \qquad (101)
\end{aligned}
$$

where $\{t - T\}$ means $(t - T)H(t - T)$ and the numbers 1 to 8 refer to the number system of Figure 3 which shows the characteristics of the flow in relation to the jet and stream. From (101) it is evident that there can be no disturbance

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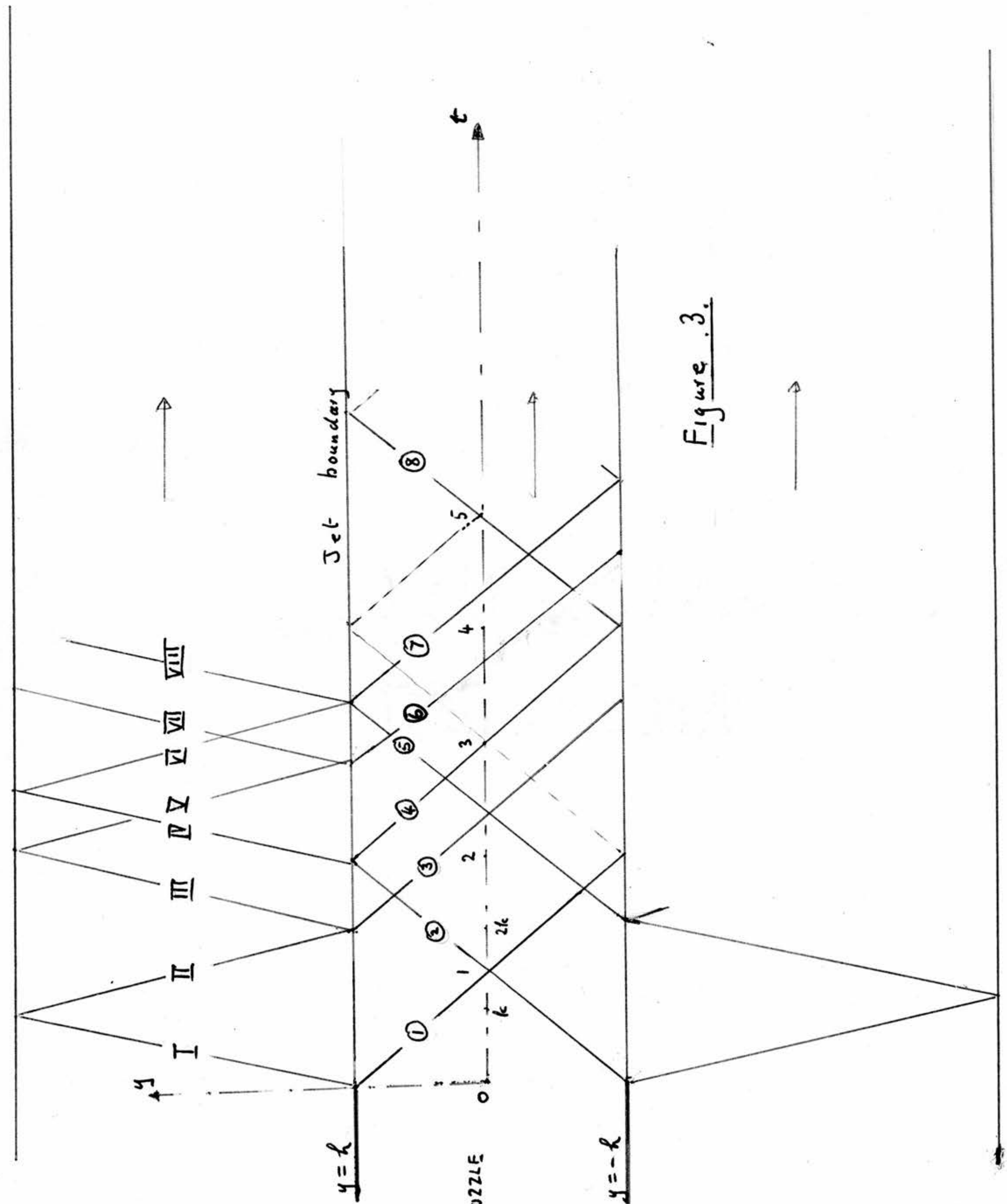

Figure 3.

upstream of the leading characteristics $t \pm y/h - 1 = 0$. A result to be expected due to the nature of the supersonic flow.

The characteristic lines represented by the first eight terms in (101) are those numbered 1 to 8 in Figure 3, which is drawn for $k<1$. The figure also shows the characteristics in the outer stream. The disturbances in the jet which are associated with the characteristics 1, 2, 4, and 8 are those due to the initial disturbance at the jet exit and the reflections at the jet boundary. The reflection coefficient, defined as the ratio of the magnitude of the reflected wave to the magnitude of the incident wave, is μ. Since μ may be positive or negative (according as m is less than or greater than unity) an expansion wave may be reflected inside the jet as an expansion or compression wave. These terms are the only ones present in the problem discussed by Pack.
In the present problem the wave disturbance inside the jet is complicated due to the presence of disturbances transmitted through the boundary from the outer stream. These disturbances first make their appearance in the jet downstream of the characteristic 3 and are associated with the characteristics 3, 5, 6, 7. These disturbances entering the jet from the outer stream are of two kinds, first, those which originate in the outer stream and then transmit into the jet and, second, those which originate in the jet, transmit into the stream and then, after reflection at the outer wall, are re-transmitted into the jet. The waves associated with characteristic 2, for

example, on reaching the jet boundary are partly reflected (associated with characteristic 4) and partly transmitted (associated with characteristic IV). This latter disturbance is reflected at the outer wall (characteristic VI) and on meeting the jet boundary again is partly reflected (characteristic VIII) and partly transmitted into the jet (characteristic 7). Thus the disturbance near the jet boundary and immediately downstream of characteristic 7 consists of the transmitted portion of waves from the outer stream and also the reflected portion of the waves associates with characteristic 5. The amplitude coefficient of the term, corresponding to the characteristic 7, appearing in equation (101) will be examined after discussion of the solution in terms of the characteristics in the outer stream.

Consider the expansion of equation (100) in the form

$$\bar{\psi}(\xi, y) = -\frac{1}{\beta\xi^2} e^{-\xi\left[k - \frac{k(H-y)}{L}\right]} \left[1 + e^{-2k(H-y)/L}\right]\left[1 + e^{-2k\xi} + e^{-4k\xi} + \cdots\right]\left[g(\xi)\right]$$

in the region $h+0 \leq y \leq H-0$, where $g(\xi)$ has already been defined. This may be written

$$\beta(1+m)\bar{\psi}(\xi, y) = \frac{1}{\xi^2} e^{-\xi[k - k(H-y)/L]} \Big[1 + e^{-2\xi k(H-y)L} - \mu e^{-2k\xi} - (1-\mu) e^{-2\xi} - \mu e^{-2\xi[k + k(H-y)/L]} - (1-\mu) e^{-2\xi[1 + k(H-y)/L]} - \mu^2 e^{-4k\xi} + (1-\mu)(1+2\mu) e^{-2(1+k)\xi} + \cdots\Big]$$

Note that $k = \beta L/h = \beta(H-h)/h$ and interpret term by term, then,

$$
\begin{aligned}
&\beta(1+m)\psi(t,y) \\
&= -\Big[\Big\{t - \beta\tfrac{y}{h} + \beta\Big\}_{\mathrm{I}} + \Big\{t + \beta\tfrac{y}{h} + \beta - 2\beta\tfrac{H}{h}\Big\}_{\mathrm{II}} \\
&\quad - \mu\Big\{t - \beta\tfrac{y}{h} + 3\beta - 2\beta\tfrac{H}{h}\Big\}_{\mathrm{III}} - (1-\mu)\Big\{t - \beta\tfrac{y}{h} + \beta - 2\Big\}_{\mathrm{IV}} \\
&\quad - \mu\Big\{t + \beta\tfrac{y}{h} + 3\beta - 4\beta\tfrac{H}{h}\Big\}_{\mathrm{V}} - (1-\mu)\Big\{t + \beta\tfrac{y}{h} + \beta - 2 - 2\beta\tfrac{H}{h}\Big\}_{\mathrm{VI}} \\
&\quad - \mu^2\Big\{t - \beta\tfrac{y}{h} + 5\beta - 4\beta\tfrac{H}{h}\Big\}_{\mathrm{VII}} + (1-\mu)(1+2\mu)\Big\{t - \beta\tfrac{y}{h} + 3\beta - 2 - 2\beta\tfrac{H}{h}\Big\}_{\mathrm{VIII}} \qquad (102) \\
&\quad + \cdots \Big]
\end{aligned}
$$

where $\{t - T\}$ means $(t - T)H(t - T)$ and the suffixes I to VIII refer to Figure 3. This shows immediately that in the upper half of the stream there is no disturbance upstream of the leading characteristic of equation $t - \beta y/h + \beta = 0$ from the edge of the orifice.

The characteristics corresponding to the first eight terms of equation (102) are drawn, and labelled I to VIII in Figure 3. The characteristics I, II, III, V and VII are those associated with the initial and reflected disturbances which originate solely within the stream. The characteristic IV is associated with the disturbance which starts in the jet and is transmitted through the jet boundary. The characteristic VIII is associated with a disturbance which starts in the lower half of the stream is transmitted through the jet and into the upper half of the stream.

A closer examination of the two equations (101) and (102) gives the reflection and transmission coefficients. The wave reflection at the outer wall $y = H$ is without loss in amplitude and with no change in the type of wave, if the wave downstream of characteristic I is an expansion wave then so is the reflected wave downstream of the characteristic II. In the stream, the amplitude of the wave reflected from the jet boundary (characteristic III) is $(-\mu)$ times the amplitude of the incident wave (characteristic II). The reflection coefficient is thus $(-\mu)$. Note that inside the jet the reflection coefficient is $(+\mu)$. This indicates that if inside the jet an expansion wave is reflected as an expansion (compression) wave then outside the jet an expansion wave is reflected, at the boundary, as a compression (expansion) wave. The transmission coefficient, defined as the ratio of the amplitudes of the transmitted wave to the incident wave, for wave transmission from the jet into the stream and from the stream into the jet may be examined. Consider the wave associated with characteristic 2 inside the jet. The portion of this disturbance which is transmitted through the boundary is associated with the characteristic IV outside the jet and the transmission coefficient is evidently $(1-\mu)/\beta$. Consider the wave associated with the characteristic II in the stream. At the jet boundary the transmitted portion is associated with the characteristic 3 inside the jet. The transmission coefficient defined as above is thus $(1+\mu)\beta$. For example, consider the waves associated with and immediately downstream of characteristic VIII in the outer stream. This disturbance is due partly to the reflection of waves associated with characteristic VI and partly to the transmitted portion of the waves associated with characteristic 5. The amplitude coefficient is therefore given by $\left(\frac{1-\mu}{\beta}\right)[-\mu] + (-1-\mu)\left[\frac{1-\mu}{\beta}\right]$ which reduces to $-(1-\mu)(1+2\mu)/\beta$. It is seen that this is just the coefficient of the appropriate term in equation (102). Likewise the disturbance associated with the characteristic 7 near the jet boundary consists partly of the reflected portion of the disturbance associated with the characteristic 5 and partly of the transmitted portion of the disturbance associated with characteristic VI. The amplitude coefficient is then $(-1-\mu)[+\mu] + \left(\frac{1-\mu}{\beta}\right)[(1+\mu)\beta]$, which reduces to $(1+\mu)(1-2\mu)$. This is just the coefficient of the appropriate term in equation (101).

The case $\mu = 0$ corresponds to the case of no reflected waves at the jet boundary. There are transmitted waves and waves reflected at the outer walls. The Figure 2(d) illustrates, with the same lettering of the characteristics as in Figure 3, the case $\mu = 0 (m = 1)$; k = 10. In this case the wave disturbance which starts at the orifice inside the jet (characteristic 2) passes across the jet and is entirely transmitted into the outer stream (characteristic IV) and after reflection at the outer wall (characteristic VI) it is retransmitted into the jet (characteristic 7). This effect is then repeated downstream.

Conclusion

In this chapter the problem of a two-dimensional supersonic jet embedded in a two-dimensional, finite supersonic stream has been discussed. It is seen that the problem may be represented as a Wiener-Hopf type problem by a simple re-arrangement of the basic equations used in Chapter I. Whilst the solution may easily be found in an infinite series form, the solution of greater interest is that formed in terms of the characteristics of the flow. This solution is complicated due to the presence of disturbances which are reflected at and transmitted through the jet boundary both from the jet and from the stream. In particular, the wave formation of the jet boundary is calculated and discussed. The case $k = 1$ is of interest since the boundary in this case exhibits a triangular waveform for all values of m. This waveform is shown in Figure 2(a). The case $m = 1$, for which there is no reflection at the jet boundary, is discussed. It is found that the boundary oscillation is periodic when k is rational. For this case, the jet expands a certain distance, remains of constant width and then contracts to its original width. This constitutes the period of the oscillation. This period steadily increases as the outer stream width increases. In the limit, the problem reduces to the one discussed by Pack; the jet then expands for a certain distance and remains of constant width thereafter.

The solution of the problem in the infinite series form is used in order to investigate the ultimate jet width. This is found to have a mean value of $\left[1 + \varepsilon\beta_1\beta_2 k/(mk+1)\right]$ times the orifice width, in the general case. As the outer stream width increases to infinity the value of the ultimate width reduces to the value found by Pack.

The solution of the problem expressed in terms of the characteristics of the flow is used to calculate the reflection and the transmission coefficients of the disturbances at the jet boundary. In particular, it is found that for $\mu\left(= \frac{1-m}{1+m}\right)$ positive (negative), an expansion wave inside the jet is reflected at the boundary as an expansion (compression) wave whereas an expansion wave in the outer stream is reflected at the jet boundary as a compression (expansion) wave.

CHAPTER III

The flow of a subsonic two-dimensional gas jet in a uniform supersonic flow

1. Formulation of the problem and the general solution

This problem is basically similar to that discussed in Chapter I in which the outer stream was subsonic and the jet was supersonic. In the problem of Chapter I the disturbances, due to the interaction between jet and stream, were transmitted both upstream and downstream in the subsonic flow. Likewise, in the present problem it is to be expected that the disturbances will be transmitted in all directions inside the jet but only downstream of the leading characteristic in the outer stream.

The formulation of the problem is as given by the basic equations (1) to (8) of Chapter I with the understanding that $M_2 > 1$ and $M_1 < 1$. Rewrite equations (7) and (8)

$$\frac{\partial^2 \phi_1}{\partial y^2} + (\beta_1')^2 \frac{\partial^2 \phi_1}{\partial x^2} = 0$$

and

$$\frac{\partial^2 \phi_2}{\partial y^2} - (\beta_2')^2 \frac{\partial^2 \phi_2}{\partial x^2} = 0,$$

where $(\beta_1')^2 = 1 - M_1^2$ and $(\beta_2')^2 = M_2^2 - 1$.

The derivation of the Wiener-Hopf type equation now follows as indicated in Chapter I if β_1 be replaced by $i\beta_1'$ and β_2 by $i\beta_2'$. Thus, equation (28) will be replaced by

$$\left(\frac{1}{i\beta_2'}\right) \bar{v}_+(\alpha, h)\, K'(\alpha) = -\frac{\varepsilon}{\alpha\sqrt{2\pi}} - G_-(\alpha, h) \qquad (103)$$

where

$$K'(\alpha) = \frac{1}{i}\left[\cot \alpha\beta_2' h - m' \coth \alpha\beta_1' h\right] \qquad (104)$$

and $$m' = \frac{\gamma_1 M_1^2 \beta_2'}{\gamma_2 M_2^2 \beta_1'} \,, \tag{105}$$

and the other symbols are given by equations (31) to (33).

The primes may now be dropped and the procedure of Chapter I adopted, the equation corresponding to (36) becomes

$$-\frac{1}{\beta_2} \bar{v}_T(\alpha, h) K_+(\alpha) + \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha} \varepsilon K_-(0)$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{[K_-(\alpha) - K_-(0)]}{\alpha} - G_-(\alpha, h) K_-(\alpha) \tag{106}$$

where, now, $K(\alpha) = \dfrac{K_+(\alpha)}{K_-(\alpha)} = \cot \alpha\beta_2 L - m \coth \alpha\beta_1 h.$

In (106) the left-side is analytic in the upper half plane $\Im m\, \alpha > 0$ whilst the right side is analytic in the overlapping lower half plane $\Im m\, \alpha < \rho$ where $0 < \rho < \pi/\beta_1 h$. By the argument as was employed in Chapter I the two sides of (106) are both equal to an integral function which is zero. The introduction of the simplifying transformations (43) and (44) leads to

$$\bar{V}_+(\xi, h) = \frac{Q_-(0)}{\xi Q_+(\xi)}$$

and $$\bar{F}(\xi) = \frac{Q_-(0)}{\xi^2 Q_+(\xi)} \,,$$

where $$Q(\xi) = \frac{Q_+(\xi)}{Q_-(\xi)} = \cot k\xi - m \coth \xi \,.$$

The perturbation potentials are given by

$$\psi(t, y) = \frac{1}{\sqrt{2\pi}} \int_C \frac{e^{-i\xi t} \cosh(\xi y/h) \bar{V}_+(\xi, h)\, d\xi}{\xi \sinh \xi} \tag{107}$$

in the region of the subsonic jet, $0 \leq y \leq h-0$, and by

$$\psi(t,y) = \frac{1}{\sqrt{2\pi}}\,\frac{\beta_1}{\beta_2}\int_C \frac{e^{-i\xi t}\cos[k\xi(H-y)/k]\,\bar{V}_+(\xi,h)\,d\xi}{\xi\,\sin k\xi} \tag{108}$$

in the region occupied by the supersonic stream, $h+0 \leq y \leq H-0$.

It is necessary to investigate the factorisation of $Q(\xi)$ which has the form $\dfrac{\tanh\xi - m\tan k\xi}{\tanh\xi\,.\,\tan k\xi}$. There are again three different factorisations according as mk is greater than, equal to, or less than unity. The factorisation is carried out in the same manner as shown in the first Chapter. Consider only the case of $mk > 1$.

Let $\xi = \sigma + i\tau$, the function $(\tanh\xi - m\tan k\xi)$ has a zero at the origin; simple real zeros given by $\xi = \pm\sigma_n$ where $n\pi < k\sigma_n < (2n+1)\pi/2$ and n = 1,2,3...; and simple imaginary zeros $\xi = \pm i\tau_n$ where $n\pi < \tau_n < (2n+1)\pi/2$ and n = 0,1,2,3... For large values of n

$$k\sigma_n \sim n\pi + \phi$$

and

$$\tau_n \sim n\pi + \theta$$

where $\tan\theta = m$ and $\tan\phi = 1/m$. The zeros of $\tanh\xi$ and $\tan k\xi$ are $\xi = \pm i n\pi$ and $\xi = \pm n\pi$ respectively, where n = 0,1,2,3... The function $Q(\xi)$ may be factorised in the form $Q_+(\xi)/Q_-(\xi)$ where

$$Q_+(\xi) = \frac{(1-mk)\left(1-\frac{i\xi}{\tau_0}\right)}{k\xi}\prod_1^{\infty}\frac{\left(1-\frac{i\xi}{\tau_n}\right)e^{\frac{i\xi}{n\pi}}\left(1-\frac{\xi^2}{\sigma_n^2}\right)}{\left(1-\frac{i\xi}{n\pi}\right)e^{\frac{i\xi}{n\pi}}\left(1-\frac{k^2\xi^2}{n^2\pi^2}\right)}$$

and $$Q_-(\xi) = \frac{1}{(1+i\xi/\gamma_0)} \prod_1^{\infty} \frac{(1+i\xi/n\pi)\, e^{-i\xi/n\pi}}{(1+i\xi/\gamma_n)\, e^{-i\xi/n\pi}}$$

The functions $Q_+(\xi)$ and $Q_-(\xi)$ are analytic and non-zero in the regions $\Im m\,\xi > 0$ and $\Im m\,\xi < \gamma_0$, respectively.

When the path of integration in (108) is closed by the infinite semicircle in the upper half plane it is found that the contribution to $\psi(t, y)$ from the semicircle is zero when $t + k(H-y)/L - k < 0$. The integrand of (108) contains no poles in the upper half plane and it is concluded that the perturbation potential $\psi(t, y)$ in the stream is zero upstream of the leading characteristic from the edge of the orifice. This result is as anticipated. The evaluation of $\psi(t, y)$, and thence of $\phi(x,y)$, for the various regions of the flow may be found making use of the Cauchy residue theorem.

The potentials in the jet are given by

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = \frac{k(x + D\beta_1 h)}{\beta_1 h(mk-1)} + \sum_1^{\infty} \frac{\cosh(\sigma_r y/h)}{\sigma_r^2 \sinh \sigma_r} \frac{[A_r \sin(\sigma_r x/\beta_1 h) + B_r \cos \sigma_r x/\beta_1 h]}{[k \operatorname{cosec}^2 k\sigma_r - m \operatorname{cosech}^2 \sigma_r]} - \sum_0^{\infty} \frac{e^{-\gamma_r x/\beta_1 h} \cos(\gamma_r y/h)}{\gamma_r^2 \sin \gamma_r} \frac{[Q_-(-i\gamma_r)]^{-1}}{[k \operatorname{cosech}^2 k\gamma_r - m \operatorname{cosec}^2 \gamma_r]}$$

for $x > 0$, and by

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = \sum_1^{\infty} \frac{e^{\frac{r\pi x}{\beta_1 h}} \cos(r\pi y/h)\, \overline{V}_+(ir\pi, h)}{r\pi\,(-1)^{r+1}}$$

for $x < 0$.

The potentials in the stream are given by

$$\frac{\phi(x,y)}{\varepsilon h\beta_1} = \frac{(x + D\beta_1 h)}{\beta_1 h(mk-1)}$$

$$+ \sum_1^\infty \frac{\cos[k\sigma_r(H-y)/L]\ [A_r \sin(\sigma_r x/\beta_1 h) + B_r \cos(\sigma_r x/\beta_1 h)]}{\sigma_r^2 \sin k\sigma_r\ [k \operatorname{cosec}^2 k\sigma_r - m \operatorname{cosech}^2 \sigma_r]}$$

$$- \sum_0^\infty \frac{e^{-\tau_r x/\beta_1 h} \cosh[k\tau_r(H-y)/L]\ [Q_-(-i\tau_r)]^{-1}}{\tau_r^2 \sinh k\tau_r\ [k \operatorname{cosech}^2 k\tau_r - m \operatorname{cosec}^2 \tau_r]}$$

for $h - y + x/\beta_2 > 0$;

and by

$$\frac{\phi(x,y)}{\varepsilon h\beta_1} = 0$$

for $h - y + x/\beta_2 < 0$.

In the above expressions D is a constant given by $iD = Q_-'(0)$ and A_r , B_r are constants defined by equation (64).

2. Behaviour of the jet boundary

The ultimate width of the jet may be obtained by an examination of the function $F(t)$ through its transform $\bar{F}(\xi)$. The asymptotic form of $F(t)$ is given by the sum of residues at the poles of $\bar{F}(\xi)$ which have the greatest imaginary part, that is from the poles $\xi = 0$ and $\xi = \pm\sigma_n$. On converting to the original variable the ultimate width is given by $h[1 + f(x)]$ where

$$\frac{f(x)}{\varepsilon\beta_1\beta_2} = -\frac{k}{(mk-1)} + \text{terms in } \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\left(\frac{\sigma_r x}{\beta_1 h}\right) .$$

Thus the jet boundary fluctuates about a mean width which is $\left[1 - \frac{\varepsilon\beta_1\beta_2 k}{(mk-1)}\right]$ times the original width. In the overpressure case $(\varepsilon > 0)$ there is an overall decrease in the jet width but, in the underpressure case $(\varepsilon < 0)$ there is an overall increase. These results are exactly opposite to those found for the same case $(mk > 1)$ in the problem of a supersonic jet in a subsonic stream (see equation (70) et seq).

An examination of the initial jet slope is undertaken by finding the asymptotic form of $\bar{F}(\zeta)$ for $|\zeta| \to \infty$ in the appropriate half-plane. As in Appendix 2 the product function

$$\prod_{1}^{\infty} \frac{(1 - i\zeta/\tau_n)\, e^{i\zeta/n\pi} \; (1 - \zeta^2/\sigma_n^2)}{(1 - i\zeta/n\pi)\, e^{i\zeta/n\pi} \; (1 - k^2\zeta^2/n^2\pi^2)}.$$

in which $k\sigma_n \sim n\pi + \phi = \pi(n+q)$ and $\tau_n \sim n\pi + \theta = \pi(n+p)$ as $n \to \infty$, is asymptotic to

$$\frac{\Gamma(1 - i\zeta/\pi)}{\Gamma(1 - i\zeta/\pi + p)} \quad \frac{\Gamma(1 - k\zeta/\pi)}{\Gamma(1 - k\zeta/\pi + q)} \quad \frac{\Gamma(1 + k\zeta/\pi)}{\Gamma(1 + k\zeta/\pi + q)}$$

The Stirling expansion formula may be applied to the Gamma functions and it is found that the original product function is asymptotic to $O(\zeta^{-(p+2q)})$. With p + q = 1/2, it is found that

$$Q_+(\zeta) \sim O(\zeta^{-(1-p)}) \quad \text{for large } |\zeta| .$$

Thus $\bar{F}(\xi) \sim O(1/\xi^{1+\rho})$ and it follows that

$$f(x) \sim O(x^\rho) \qquad \text{for} \quad x \to 0+$$

where $0 < \rho = \frac{1}{\pi}\tan^{-1} m < \frac{1}{2}$.

It is seen that the initial shape of the boundary lies between the parabola $y = h[1 + O(\sqrt{x})]$ and the straight line, parallel to the jet exit, of equation $y = h[1 + O(1)]$. The initial slope is infinite being given by $f'(x) = O(x^{-1+\rho})$ These results for the subsonic jet in a supersonic stream for $mk > 1$ are very similar to those already obtained in the case of a supersonic jet in a subsonic stream for $mk < 1$. Similarly it may be shown that the cases $mk \leq 1$ for the present problem lead to solutions which are similar to those obtained in Chapter I for the cases $mk \geq 1$. The location of the roots shows that the case mk = 1 is still singular.

CHAPTER IV

The flow of a subsonic two-dimensional gas jet in a uniform subsonic stream

1. Formulation of the problem and the general solution

The problem of a subsonic two-dimensional compressible gas jet emerging from an orifice into a region at rest has been discussed by Reynolds (1) and subsequently by Chaplygin (3) who used the theory of functions of a complex variable. This latter work provided a formal solution to the problem of a jet issuing through a slit in an infinite straight wall in terms of hypergeometric functions. This work was generalised by Jacobs (4) who considered the problem with the walls of the reservoir inclined at an angle to the jet axis. Later work has been primarily concerned with finding approximate forms of the formal solutions provided by Chaplygin and Jacobs.

So far as is known there have been no theoretical investigations into the problem of a subsonic jet issuing into a subsonic stream of finite or infinite width. By use of the linearised theory it will be shown that a solution, in series form, can be found by employing the Wiener-Hopf technique.

The notation of Chapter I is employed; the boundary conditions are given by equations (3) to (6) and the field equations by (7) and (8) it being noted that both M_1 and M_2 are less than unity. The analysis of Chapter I may be taken to apply to the present problem if β_1 be replaced by $i\beta_1'$ where $(\beta_1')^2 = 1 - M_1^2$. This leads to the equation

$$\frac{d\bar{\phi}_+(\alpha,h)}{dy}\left[\frac{\coth\alpha\beta_2(H-h)}{\beta_2}+\ell^2\frac{\coth\alpha\beta_1' h}{\beta_1'}\right]$$

$$=-\frac{\varepsilon}{\alpha\sqrt{2\pi}}-\alpha\left[\bar{\phi}_-(\alpha,h+0)-\ell^2\bar{\phi}_-(\alpha,h-0)\right].$$

The Wiener-Hopf type equation corresponding to (28) becomes, after dropping the primes,

$$\frac{\bar{v}_+(\alpha,h)K(\alpha)}{\beta_2}=-\frac{\varepsilon}{\alpha\sqrt{2\pi}}-\bar{G}_-(\alpha,h)$$

in which $K(\alpha)=\coth\alpha\beta_2 L+m\coth\alpha\beta_1 h$

and the other functions are defined by the equations (30) to (33). It is again necessary to investigate the regions of convergence of the transforms in the above equation. First, $\bar{v}_+(\alpha,h)$ is analytic in the region $\Im m\,\alpha>0$; second, $K(\alpha)$ has poles at the zeros of $\tanh\alpha\beta_2 L$ and $\tanh\alpha\beta_1 h$ and hence $K(\alpha)$ is analytic in the strip $0<\Im m\,\alpha<d_1$ where d_1 is $\mathrm{Min}\left(\frac{\pi}{\beta_2 L}\,;\,\frac{\pi}{\beta_1 h}\right)$. It may be noted that if $k\left(=\frac{\beta_2 L}{\beta_1 h}\right)>1$ then $d_1=\frac{\pi}{\beta_2 L}$; if $k<1$ then $d_1=\frac{\pi}{\beta_1 h}$; and if $k = 1$ the two values coincide. Third, by considering the asymptotic forms of $\phi_-(x,h+0)$ and $\phi_-(x,h-0)$ as $x\to-\infty$, $\bar{G}_-(\alpha,h)$ is analytic in the lower half plane $\Im m\,\alpha<d_2$ where $d_2=\mathrm{Min}\left(\frac{\pi}{\beta_2 h}\,;\,\frac{\pi}{\beta_1 L}\right)$. Fourth, the transform $1/\alpha$ is analytic in the upper half plane $\Im m\,\alpha>0$.

It is also necessary to investigate the factorisation of the function $K(\alpha)$. It may be shown (26 Pg 324) that $K(\alpha)$

has all of its zeros simple and imaginary and the factorisation into $K_+(\alpha) / K_-(\alpha)$ can be easily made. The Wiener-Hopf type equation is given by (36)

$$\frac{\bar{v}_+(\alpha,h)\,K_+(\alpha)}{\beta_2} + \frac{\varepsilon K_-(0)}{\alpha\sqrt{2\pi}} = -\frac{[K_-(\alpha) - K_-(0)]}{\alpha\sqrt{2\pi}} - G_-(\alpha, h)\,K_-(\alpha)$$

in which the right side is analytic and non-zero in an upper half plane and the left side is analytic and non-zero in an overlapping lower half plane. As previously shown the two sides are equal to an integral function. It may be argued on physical grounds that the integral function is identically zero. It may be noted that it is possible to show that the integral function is zero, at least in the case when k is rational, by an analysis similar to that of Appendix 2.

The transformations (43) and (44) may again be applied to lead to the equations (45) to (48) but (49) is replaced by

$$Q(\xi) = \frac{Q_+(\xi)}{Q_-(\xi)} = \coth k\xi + m \coth \xi \,. \qquad (109)$$

Consider the factorisation of $Q(\xi)$. Now

$$Q(\xi) = \frac{\tanh \xi + m \tanh k\xi}{\tanh \xi \,\tanh k\xi}$$

and $(\tanh \xi + m \tanh k\xi)$ has zeros given by $\xi = 0$, and $\xi = \pm i\gamma_n$ where γ_n are the real and simple roots of $\tan\theta + m\tan k\theta = 0$. This set of roots includes the common roots of $\cos\theta$ and $\cos k\theta$ which may occur if k is rational. The factorisation of $Q(\xi)$ may now be performed to give

$$Q_+(\xi) = \frac{(1+mk)\, e^{\chi(\xi)}}{k\xi} \prod_1^\infty \frac{(1 - i\xi/\tau_n)\, e^{i\xi/\tau_n}}{(1 - i\xi/n\pi)\, e^{i\xi/n\pi}\; (1 - ik\xi/n\pi)\, e^{ik\xi/n\pi}} \tag{110}$$

and

$$Q_-(\xi) = e^{\chi(\xi)} \prod_1^\infty \frac{(1 + i\xi/n\pi)\, e^{-i\xi/n\pi}\; (1 + ik\xi/n\pi)\, e^{-ik\xi/n\pi}}{(1 + i\xi/\tau_n)\, e^{-i\xi/\tau_n}} . \tag{111}$$

where $\chi(\xi)$ is a function to be chosen so that $Q_+(\xi)$ and $Q_-(\xi)$ are of algebraic growth when $|\xi|$ is infinitely large. $Q_+(\xi)$ is analytic and non-zero in the upper half plane $\Im m\,\xi > 0$, and $Q_-(\xi)$ is analytic and non-zero in the lower half plane $\Im m\,\xi < \tau_0$.

The perturbation potentials may now be found and the problem solved by calculating the function $\psi(t, y)$ of equation (44). Corresponding to the equations (61) and (62) there are the two equations

$$\psi(t,y) = \frac{1}{\sqrt{2\pi}} \int_C \frac{e^{-i\xi t} \cosh(\xi y/h)\; \bar{V}_+(\xi, h)\; d\xi}{\xi \sinh \xi} \tag{112}$$

$$0 \leq y \leq h-0,$$

and

$$\psi(t,y) = -\frac{1}{\sqrt{2\pi}} \frac{\beta_1}{\beta_2} \int_C \frac{e^{-i\xi t} \cosh[k\xi(H-y)/h]\; \bar{V}_+(\xi, h)\; d\xi}{\xi \sinh k\xi} \tag{113}$$

$$h+0 \leq y \leq H-0$$

where $\bar{V}_+(\xi,h) = -\dfrac{Q_-(0)}{\xi\, Q_+(\xi)}$

and $Q_+(\xi)$ is given by (110).
From (111) it is seen that $Q_-(0) = e^{\chi(0)}$. It may be shown, at least for k rational, that $\chi(\xi)$ is linear in ξ and that $\chi(0) = 0$. It follows that $Q_-(0) = 1$.

2. The perturbation potentials

Consider the evaluation of (112) for the region inside the jet. The integrand has poles given by $\xi = 0$ and $\xi = -i\tau_n$ and $\xi = +in\pi$. If the contour be closed by an infinite semicircle in the lower half plane then only the poles $\xi = 0$ and $\xi = -i\tau_n$ are enclosed. Write $-i\psi(t,y)/\sqrt{2\pi} = R_1 + \Sigma R_n$ where R_1 and R_n are the residues at the poles $\xi = 0$ and $\xi = -i\tau_n$ respectively. At $\xi = 0$ the integrand takes the form

$$\frac{\left[1 - i\xi t + o(\xi^2)\right]\left[1 + o(\xi^2)\right]\left[1 - \xi Q_-'(0) + o(\xi^2)\right]}{\xi^2\left[\frac{1}{k} + m\right]}$$

The term R_1 is thus $\dfrac{-ik\left(t - iQ_-'(0)\right)}{(1+mk)}$

The term R_n is

$$-\frac{i\, e^{-\tau_n t}\cos(\tau_n y/h)\left[Q_-(-i\tau_n)\right]^{-1}}{\tau_n^2 \sin\tau_n\left[k\,\mathrm{cosec}^2 k\tau_n + m\,\mathrm{cosec}^2\tau_n\right]}$$

The function $\psi(t,y)$ is given by

$$\frac{\psi(t,y)}{\sqrt{2\pi}} = \frac{k\left(t - iQ_-'(0)\right)}{(1+mk)} + \sum_{1}^{\infty}\frac{e^{-t\tau_n}\cos(\tau_n y/h)\left[Q_-(-i\tau_n)\right]^{-1}}{\tau_n^2\sin\tau_n\left(k\,\mathrm{cosec}^2 k\tau_n + m\,\mathrm{cosec}^2\tau_n\right)}$$

in the region $t > 0, \; 0 \leq y \leq h-0.$

Closure of the contour in the upper half plane encloses the poles $\zeta = +in\pi$ and thus

$$\frac{\psi(t,y)}{\sqrt{2\pi}} = \sum_{1}^{\infty} \frac{e^{n\pi t} \cos(n\pi y/h) \, \bar{V}_+(in\pi, h)}{n\pi \cos n\pi}$$

in the region $t < 0, \; 0 \leq y \leq h-0.$

Similarly, the equation (113) may be evaluated to find $\psi(t,y)$ in the region occupied by the stream. Transforming into the original variables of the problem leads to

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = \frac{k[x - i\beta_1 h \, Q_-'(0)]}{\beta_1 h(mk+1)} + \sum_{1}^{\infty} \frac{e^{-\tau_n x/\beta_1 h} \cos(\tau_n y/h) \, [Q_-(-i\tau_n)]^{-1}}{\tau_n^2 \sin \tau_n \, [k \operatorname{cosec}^2 k\tau_n + m \operatorname{cosec}^2 \tau_n]} \qquad (114)$$

in the region $x > 0, \; 0 \leq y \leq h-0,$

and

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = \sum_{1}^{\infty} \frac{e^{\frac{n\pi x}{\beta_1 h}} \cos(n\pi y/h) \, \bar{V}_+(in\pi, h)}{n\pi \cos n\pi} \qquad (115)$$

in the region $x < 0, \; 0 \leq y \leq h-0;$

together with

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} = -\frac{[x - i\beta_1 h \, Q_-'(0)]}{\beta_1 h(mk+1)} - \sum_{1}^{\infty} \frac{e^{-\tau_n x/\beta_1 h} \cos[k\tau_n(H-y)/L] \, [Q_-(-i\tau_n)]^{-1}}{\tau_n^2 \sin k\tau_n \, [k \operatorname{cosec}^2 k\tau_n + m \operatorname{cosec}^2 \tau_n]} \qquad (116)$$

in the region $x>0, \; h+0 \leq y \leq H-0,$

and

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = -\frac{L}{h}\sum_1^{\infty} \frac{e^{n\pi x/\beta_2 L}\cos[n\pi(H-h)/L]\, \bar{V}_+\left(\frac{in\pi}{L}, h\right)}{n\pi \cos n\pi} \qquad (117)$$

in the region $x<0, \; h+0 \leq y \leq H-0.$

It is observed that these solutions differ from the solutions of the previous three cases in that there are no periodic (or almost periodic) terms present. The terms under the summation signs decay exponentially as the distance from the jet orifice increases both upstream and down. Downstream of the orifice the jet quickly settles down into complete uniformity a result in agreement with the practical observations made on a subsonic jet emerging from an orifice into a receiver under small excess pressure.

3. <u>The behaviour of the jet boundary</u>

Consider now the shape of the jet boundary. The equation of the boundary is written in the form $y = h[1+f(x)]$ for $x>0$ and the function F(t) of equation (44) is introduced. This function is given by the inverse transform of

$$\bar{F}(\xi) = -\frac{1}{\xi^2 Q_+(\xi)}$$

where $Q_+(\xi)$ is given by (110). Thus

$$F(t) = -\frac{1}{\sqrt{2\pi}}\int_C \frac{e^{-i\xi t}\, d\xi}{\xi^2(\coth k\xi + m\coth\xi)\, Q_-(\xi)}$$

$$= i\sqrt{2\pi}\left[\frac{k}{(1+mk)} - \sum_1^\infty \frac{e^{-\tau_r t}\,[Q_-(-i\tau_r)]^{-1}}{\tau_r^2\,[k\,\mathrm{cosec}^2 k\tau_r + m\,\mathrm{cosec}^2\tau_r]}\right]$$

which leads to

$$\frac{F(x)}{\varepsilon\beta_1\beta_2} = -\frac{k}{(mk+1)} + \sum_1^\infty \frac{e^{-\tau_r x/\beta_1 h}\,[Q_-(-i\tau_r)]^{-1}}{\tau_r^2\,[k\,\mathrm{cosec}^2 k\tau_r + m\,\mathrm{cosec}^2\tau_r]} \qquad (118)$$

for $x > 0$.

Equation (118) shows that far downstream of the jet exit $(x \to +\infty)$ the width of the jet boundary is given by $y = h\left[1 - \frac{\varepsilon\beta_1\beta_2 k}{mk+1}\right]$ which shows that there is a small decrease in the jet width for the case $\varepsilon > 0$. This result corresponds to the vena contracta observed in a jet which is emitted into a region at rest. However, for the case $\varepsilon < 0$, when the jet emerges into a stream of greater pressure there is a small expansion in the jet width. The change in the jet width takes place without restriction on the value of $k \left(= \frac{\beta_2 L}{\beta_1 h}\right)$ and for a stream of infinite width (k infinite) the ultimate change in the width of the jet is just $\frac{\varepsilon\beta_1\beta_2 h}{m}$.

It may be observed that the slope of the jet boundary is given by $$p'(x) = -\frac{\varepsilon\beta_2}{h}\sum_1^\infty \frac{e^{-\gamma_r x/\beta_1 h}\left[Q_-(-i\gamma_r)\right]^{-1}}{\gamma_r\left[k\,\mathrm{cosec}^2 k\gamma_r + m\,\mathrm{cosec}^2\gamma_r\right]}$$

and is negative (for $\varepsilon > 0$) and tends to zero as $x \to \infty$.

4. Conclusions

In this Chapter the solution to the problem of a subsonic gas jet embedded in a two-dimensional subsonic stream has been found. The absence of any supersonic flow makes the solution differ from those of the previous Chapters. The perturbation potentials contain no terms, in the x variable, of a periodic (or almost periodic) nature but they do contain those terms which decay exponentially as the distance (upstream or downstream) measured from the orifice increases. The jet is shown to have a monotonic structure and it settles down rapidly to a uniform state. For the overpressure (underpressure) case $\varepsilon > 0$ ($\varepsilon < 0$) the jet width steadily decreases (increases) to an ultimate width given by $y = h\left[1 - \frac{\varepsilon\beta_1\beta_2 k}{mk+1}\right]$. It may be noted that this result is just the opposite to that found for the supersonic jet embedded in a supersonic stream.

5. Remarks on the ultimate behaviour of the general two-dimensional compound jet

It is of interest to show that the ultimate behaviour of the jet boundary, in each of the four problems discussed in Part I, can be anticipated on physical grounds. Consider a stream tube in a steady flow. Let ω , ρ , p and q be the cross section, density, pressure and velocity, respectively, at a distance x along the tube measured from a datum position (the line x = 0 in the compound jet problems). There are three basic equations

(a) Flux of mass $\omega \rho q = \text{constant},$

(b) Bernoulli's theorem $\frac{1}{2} q^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \text{constant};$

(c) the gas law $p \rho^{-\gamma} = \text{constant}.$

By differentiating (a), (b) and (c) with respect to x and using the relationship $a^2 = \gamma p/\rho$, where a is the local velocity of sound, it may be shown that

$$\frac{(1-M^2)}{\rho q^2} \frac{dp}{dx} = \frac{1}{\omega} \frac{d\omega}{dx} \qquad (A)$$

where M is the Mach number at the point under consideration. Consider the compound jet for the overpressure case $\varepsilon > 0$, i.e. $P_1 > P_2$. Far downstream, the pressures in both jet and stream become equal and hence the pressure in the jet must fall and the pressure in the stream must rise as x increases. Thus $\frac{dp}{dx}$ is negative in the jet and positive in the stream (see Figure 4).

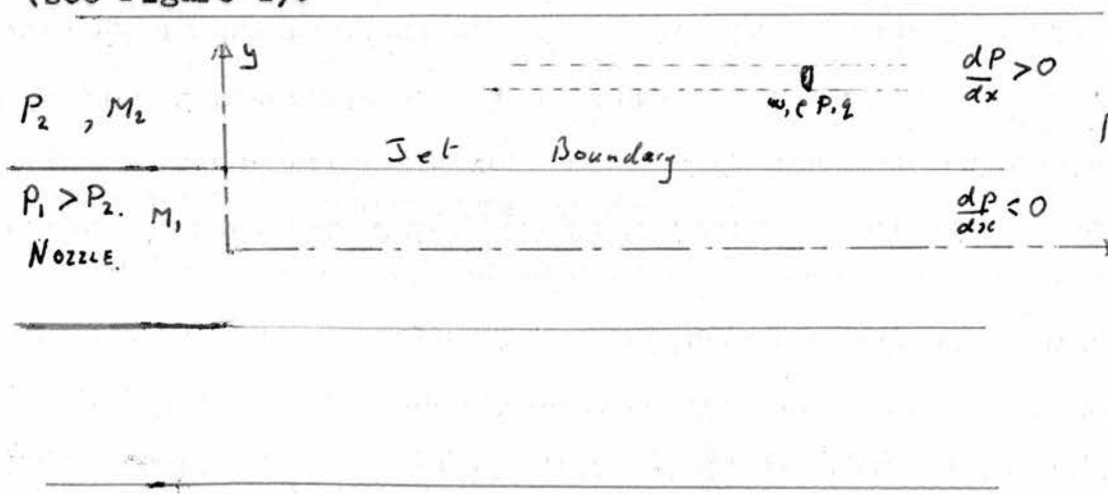

Figure. 4.

There are four possible combinations of subsonic and supersonic flows. Consider each in turn.

5.1 <u>Subsonic jet in a subsonic stream (Chapter IV)</u>

In this case $(1 - M^2)$ is positive in both flows and, from equation (A) above, it follows that $\frac{d\omega}{dx}$ is negative in the jet and positive in the stream. Thus the stream tubes in the jet will contract and those in the stream will expand. The overall effect is to make the jet boundary contract. This contraction has been shown (see Equation (118)) to be $\frac{\varepsilon h \beta_1 \beta_2 k}{(mk+1)}$.

5.2 <u>Supersonic jet in a supersonic stream (Chapter II)</u>

In this case $(1 - M^2)$ is negative in both flows and hence the stream tubes in the jet will expand and those in the stream will contract. The overall effect on the jet boundary is an expansion of its width. This expansion has been shown (see equation (95)) to be $\frac{\varepsilon h \beta_1 \beta_2 k}{(mk+1)}$

5.3 <u>Supersonic jet in a subsonic stream (Chapter I)</u>

In this case $(1 - M^2)$ is negative in the jet but positive in the stream. The result (A) shows that $\frac{dw}{dx}$ is positive in both the jet and the stream. Thus the stream tubes, in both flows, expand. The boundary of the jet may therefore undergo either an overall expansion or an overall contraction. It has been shown (see equation (70)) that the magnitude of the displacement is $\frac{\varepsilon h \beta_1 \beta_2 k}{(mk-1)}$ and that the jet expands or contracts according as $mk \left(= \frac{\gamma_1 M_1^2 \beta_2^2 L}{\gamma_2 M_2^2 \beta_1^2 h}\right)$ is greater than or less than unity.

5.4 Subsonic jet in a supersonic stream (Chapter III)

In this case $(1 - M^2)$ is positive in the jet and negative in the stream. The stream tubes, in both flows, will contract and the jet boundary may expand or contract. It has been shown (see Section 2, Chapter III) that the magnitude of the displacement is $\frac{\varepsilon h \beta_1 \beta_2 k}{(mk-1)}$ and that the jet contracts or expands according as mk is greater than or less than unity.

CHAPTER V

The flow of an axially symmetrical supersonic jet in a uniform axially symmetrical subsonic stream

The first attempt to solve this problem with an infinite outer stream was undertaken by Pai[20] who concluded that, as in the two dimensional case, the flow pattern in the jet was almost periodic. The solutions presented by Pai were incorrect since the boundary conditions of the problem were not fully specified, no account being taken of the fact that the disturbances in the stream would spread upstream. The solutions he obtained were therefore true only far downstream of the jet exit.

The problem of an axially symmetrical supersonic jet issuing into an axially symmetrical subsonic stream of finite width will now be considered. When the boundary conditions are completely specified, with the restriction of linearised theory, the problem may be reduced to one of the Wiener-Hopf type. It will be shown that the problem is capable of solution by employing the technique of the previous chapters and that the solution, obtained in a series form, includes the solution of Pai. The basic Wiener-Hopf type equation which will be obtained is naturally more complicated than that of the two-dimensional case due to the fact that the trigonometric (and hyperbolic) functions of the latter case are replaced by Bessel functions. It will be shown, however, that the mathematical procedure is almost identical with that of the previously solved problems.

The formulation of the problem proceeds as in Chapter I.

Let the axis of x be the line of symmetry and let the ordinate y now correspond to the perpendicular distance measured from this axis. Let the radius of the nozzle be h and the radius of the outer stream be H. The boundary conditions of the problems are specified by equation (9) to (13) but the equations satisfied by the potentials differ from the previous equations (14) and (15). In the established notation the problem may be restated as follows.

$$\frac{\partial \phi_+(x, h+0)}{\partial y} = \frac{\partial \phi_+(x, h-0)}{\partial y} = v_+(x, h) \quad , \tag{9}$$

$$\frac{\partial \phi_-(x, h+0)}{\partial y} = \frac{\partial \phi_-(x, h-0)}{\partial y} = 0 \quad , \tag{10}$$

$$\frac{\partial \phi(x, 0)}{\partial y} = 0 \quad , \tag{11}$$

$$\frac{\partial \phi(x, H-0)}{\partial y} = 0 \quad , \tag{12}$$

and

$$\frac{\partial \phi_+(x, h+0)}{\partial x} - \ell^2 \frac{\partial \phi_+(x, h-0)}{\partial x} = -\varepsilon \quad , \tag{13}$$

together with the equations satisfied by the potentials

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{1}{y}\frac{\partial \phi}{\partial y} - \beta_1^2 \frac{\partial^2 \phi}{\partial x^2} = 0, \tag{119}$$
$$0 \leq y \leq h-0, \quad -\infty < x < +\infty .$$

and

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{1}{y}\frac{\partial \phi}{\partial y} + \beta_2^2 \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{120}$$
$$h+0 \leq H-0, \quad -\infty < x < +\infty \quad ,$$

where $\beta_1^2 = M_1^2 - 1 \ (>0)$ and $\beta_2^2 = 1 - M_2^2 \ (<1)$

The Fourier transform theorem is applied to the equations (119) and (120) to give the Bessel type differential equations

$$\frac{d^2\bar{\phi}(\alpha,y)}{dy^2} + \frac{1}{y}\frac{d\bar{\phi}(\alpha,y)}{dy} + \alpha^2\beta_1^2\bar{\phi}(\alpha,y) = 0 \tag{121}$$

and

$$\frac{d^2\bar{\phi}(\alpha,y)}{dy^2} + \frac{1}{y}\frac{d\bar{\phi}(\alpha,y)}{dy} - \alpha^2\beta_2^2\bar{\phi}(\alpha,y) = 0 \tag{122}$$

respectively.

The general solution of (121) is

$$\bar{\phi}(\alpha,y) = A\,J_0(\alpha\beta_1 y) + B\,Y_0(\alpha\beta_1 y)$$

where $J_0(z)$ and $Y_0(z)$ are the Bessel functions of zero order and of the first and second kind respectively; and A and B are functions of α only. The particular solution of (121) appropriate to the boundary conditions (9), (10) and (11) takes the form

$$\bar{\phi}(\alpha,y) = \bar{\phi}_+(\alpha,y) + \bar{\phi}_-(\alpha,y) = -\frac{\bar{v}_+(\alpha,h)\,J_0(\alpha\beta_1 y)}{\alpha\beta_1\,J_1(\alpha\beta_1 h)} \tag{123}$$

The general solution of (122) is

$$\bar{\phi}(\alpha,y) = C\,I_0(\alpha\beta_2 y) + D\,K_0(\alpha\beta_2 y)$$

where $I_0(z)$ and $K_0(z)$ are the zero order Bessel functions of imaginary argument and of the first and second kind respectively.

The solution of (122) appropriate to the boundary conditions

(9), (10) and (12) is

$$\bar{\phi}(\alpha, y) = \bar{\phi}_+(\alpha, y) + \bar{\phi}_-(\alpha, y)$$

$$= -\frac{\bar{\vartheta}_+(\alpha, h)}{\alpha\beta_2}\left[\frac{I_1(\alpha\beta_2 H)\,K_0(\alpha\beta_2 y) + K_1(\alpha\beta_2 H)\,I_0(\alpha\beta_2 y)}{I_1(\alpha\beta_2 H)\,K_1(\alpha\beta_2 h) - K_1(\alpha\beta_2 H)\,I_1(\alpha\beta_2 h)}\right]. \qquad (124)$$

Satisfaction of the final boundary condition (13) leads to the equation

$$\frac{\bar{\vartheta}_+(\alpha, h)\,K(\alpha)}{\beta_2} = -\frac{\varepsilon}{\alpha\sqrt{2\pi}} - G_-(\alpha, h)$$

where

$$K(\alpha) = \frac{I_1(\alpha\beta_2 H)\,K_0(\alpha\beta_2 h) + K_1(\alpha\beta_2 H)\,I_0(\alpha\beta_2 h)}{I_1(\alpha\beta_2 H)\,K_1(\alpha\beta_2 h) - K_1(\alpha\beta_2 H)\,I_1(\alpha\beta_2 h)} - m\frac{J_0(\alpha\beta_1 h)}{J_1(\alpha\beta_1 h)}.$$

The investigation into the ranges of convergence of the transforms shows that $\bar{\vartheta}_+(\alpha, h)$ and $\varepsilon/\alpha\sqrt{2\pi}$ are analytic in the upper half plane $\mathcal{I}m\,\alpha > 0$ and that the function $G_-(\alpha, h)$ is analytic in the lower half plane $\mathcal{I}m\,\alpha < \eta/\beta_2$ where η is the smallest non-zero positive root of $J_1(\eta h)\,Y_1(\eta H) - Y_1(\eta h)\,J_1(\eta H) = 0$. Inspection of $K(\alpha)$ shows that it has poles on the real axis at the zeros of $J_1(\alpha\beta_1 h)$ and poles on the imaginary axis at the zeros of $I_1(\alpha\beta_2 H)\,K_1(\alpha\beta_2 h) - K_1(\alpha\beta_2 H)\,I_1(\alpha\beta_2 h)$. It may be shown that there are no other poles or singularities and that $K(\alpha)$ is analytic in the strip $0 < \mathcal{I}m\,\alpha < \eta/\beta_2$. There is thus a common strip of convergence and it remains to show that $K(\alpha)$ can be factorised in the form $K_+(\alpha)/K_-(\alpha)$

and the equation rearranged in such a way that the two sides are analytic and non-zero in an upper half plane and an overlapping lower half plane, respectively. It is shown in Appendix 3 that $K(\alpha)$ has no complex zeros but only simple ones which lie on the real and imaginary axes. There are no branch points present because, on inserting the expansions for K_0 and K_1 into $K(\alpha)$, the logarithmic terms cancel. The factorisation of $K(\alpha)$ may then be performed by infinite product factorisation and the equation rearranged in the appropriate manner. The problem is thus capable of solution providing that the integral function $E(\alpha)$ of equations (37) and (38) can be found. This function is shown to be identically zero (Appendix 4).

With the transformations (43) and (44) the following formal equations may be obtained for the perturbation potentials and the boundary displacement:

$$\psi(t,y) = \frac{1}{\sqrt{2\pi}} \int_c \frac{e^{-i\xi t}\, J_0(\xi y/h)\; Q_-(0)\, d\xi}{\xi^2\, J_1(\xi)\; Q_+(\xi)}, \qquad 0 \le y \le h-0; \tag{125}$$

$$\psi(t,y) = \frac{1}{\beta\sqrt{2\pi}} \int_c \frac{e^{-i\xi t}\left[I_1(k_1\xi)\, K_0(\frac{\beta\xi y}{h}) + K_1(k_1\xi)\, I_0(\frac{\beta\xi y}{h})\right] Q_-(0)\, d\xi}{\xi^2\left[I_1(k_1\xi)\, K_1(\beta\xi) - K_1(k_1\xi)\, I_1(\beta\xi)\right] Q_+(\xi)}, \qquad h+0 \le y \le H-0, \tag{126}$$

and

$$F(t) = -\frac{1}{\sqrt{2\pi}} \int_c \frac{e^{-i\xi t}\; Q_-(0)\; d\xi}{\xi^2\, Q_+(\xi)} \tag{127}$$

where

$$Q(\zeta) = \frac{Q_+(\zeta)}{Q_-(\zeta)} = \frac{I_1(k_1\zeta)\,K_0(\beta\zeta) + K_1(k_1\zeta)\,I_0(\beta\zeta)}{I_1(k_1\zeta)\,K_1(\beta\zeta) - K_1(k_1\zeta)\,I_1(\beta\zeta)} - m\frac{J_0(\zeta)}{J_1(\zeta)} \quad (128)$$

and $\beta = \beta_2/\beta_1$, $k_1 = \beta H/h$, (129)

and C is a straight line drawn from $-\infty$ to $+\infty$ in the strip of convergence.

It is necessary to consider the factorisation of $Q(\zeta)$.

When ζ is real, the function $\frac{J_0(\zeta)}{J_1(\zeta)}$ behaves like a cotangent with zeros at the zeros of $J_0(\zeta)$ and asymptotes at the zeros of $J_1(\zeta)$ (the zeros of $J_0(\zeta)$ and $J_1(\zeta)$ being interlaced), also the function $\frac{I_1(k_1\zeta)\,K_0(\beta\zeta) + K_1(k_1\zeta)\,I_0(\beta\zeta)}{I_1(k_1\zeta)\,K_1(\beta\zeta) - K_1(k_1\zeta)\,I_1(\beta\zeta)}$ behaves like a hyperbolic cotangent being infinite at $\zeta = 0$ and decreasing steadily towards unity as ζ increases towards infinity. For small ζ the function $Q(\zeta)$ has the form

$$\frac{2}{\zeta}\left[\frac{\beta}{k_1^2-\beta^2} - m\right] + O(\zeta)$$

and for large (real) ζ has the asymptotic form

$$\left[\frac{e^{k_1\zeta}e^{-\beta\zeta} + e^{-k_1\zeta}e^{\beta\zeta}}{e^{k_1\zeta}e^{-\beta\zeta} - e^{-k_1\zeta}e^{\beta\zeta}} - m\frac{\cos(\zeta-\pi/4)}{\sin(\zeta-\pi/4)}\right]\left[1 + O\left(\frac{1}{\zeta}\right)\right]$$

$$\sim \coth(k_1-\beta)\zeta - m\cot(\zeta-\pi/4) = \coth k\zeta - m\cot\left(\zeta-\frac{\pi}{4}\right),$$

The similarity between the $Q(\xi)$ of the present problem and that of the corresponding two-dimensional problem indicates that the form of the solutions in each problem will, in many respects, be similar. From the form of $Q(\xi)$ for small ξ it follows that the position of the zeros, and hence the factorisation, is different according as $m\lambda > 1$, $m\lambda = 1$ or $m\lambda < 1$ where $\lambda = \frac{k_1^2 - \beta^2}{\beta} = \beta\left[\frac{H^2}{h^2} - 1\right]$. This corresponds exactly with the two-dimensional case in which the factorisation is different according as $mk > 1$, $mk = 1$ or $mk < 1$ where $k = \beta\left[\frac{H}{h} - 1\right]$. The real zeros of $Q(\xi)$ are given by $\xi = \pm\sigma_n$ where $j_{1,n} < \sigma_n < j_{0,n+1}$ and $j_{r,n}$ $(n>0)$ are the roots of $J_r(z) = 0$ arranged in ascending order of magnitude and $j_{1,0} = 0$. For $m\lambda > 1$, n takes the values 0,1,2... and for $m\lambda < 1$, n takes the values 1,2,3... The imaginary zeros of $Q(\xi)$ are given by $\xi = \pm i\tau_n$ where τ_n is a real zero of

$$\frac{J_1(k_1\tau)\,Y_0(\beta\tau) - Y_1(k_1\tau)\,J_0(\beta\tau)}{Y_1(k_1\tau)\,J_1(\beta\tau) - J_1(k_1\tau)\,Y_1(\beta\tau)} - m\frac{I_0(\tau)}{I_1(\tau)} \tag{130}$$

It may be noted that for small τ the function (130) has the form already quoted but that for large τ it takes the asymptotic form $\left[\cot k\tau - m\right]\left[1 + O\left(\frac{1}{\tau}\right)\right]$. The function $I_0(\tau)/I_1(\tau)$ does, however, behave like an hyperbolic cotangent. From a sketch of the function (130) plotted against τ it may be seen that $K_{1,n} < \tau_n < K_{0,n+1}$, where $K_{r,n}$ are the (non-zero) roots of $J_1(k_1\tau)\,Y_r(\beta\tau) - Y_1(k_1\tau)\,J_r(\beta\tau) = 0$

arranged in ascending order of magnitude. In this arrangement when $m\lambda > 1$, n takes the values 1,2,3... but when $m\lambda < 1$, n takes the values 0,1,2,3... with the understanding that $K_{1,0} = 0$. From the asymptotic forms of (128), $\{$with ξ real$\}$, and of (130), with τ real (ξ imaginary), and of $J_1(k_1\tau)Y_r(\beta\tau) - Y_1(k_1\tau)J_r(\beta\tau)$; $J_0(\sigma)$ and $J_1(\sigma)$ the following results may be obtained:

$$\left.\begin{aligned} \sigma_n &\sim n\pi + \theta + \tfrac{\pi}{4} = \pi(n + p + \tfrac{1}{4}), \\ k\tau_n &\sim n\pi + \phi = \pi(n+q), \\ K_{0,n} &\sim (2n+1)\pi/2k, \\ K_{1,n} &\sim n\pi/k, \\ j_{0,n} &\sim \pi(n - 1/4), \\ j_{1,n} &\sim \pi(n + \tfrac{1}{4}), \end{aligned}\right\} \qquad (131)$$

where $\theta = \tan^{-1} m = \pi p$ and $\phi = \tan^{-1}(1/m) = \pi/2 - \theta = \pi q$.

The poles of $Q(\xi)$ are simple and occur at $\xi = \pm i K_{1,n}$ and $\xi = \pm j_{1,n}$. The function $Q(\xi)$ may now be factorised by writing it in infinite product form. Consider only the case of $m\lambda > 1$. Now

$$Q(\xi) = \frac{2(1-m\lambda)}{\lambda\xi}(1 - \xi^2/\sigma_0^2)\prod_1^\infty \frac{(1 + \xi^2/\tau_n^2)(1 - \xi^2/\sigma_n^2)}{(1 + \xi^2/K_{1,n}^2)(1 - \xi^2/j_{1,n}^2)}$$

and hence

$$Q_+(\xi) = \frac{2(1-m\lambda)}{\lambda\xi}(1-\xi^2/\sigma_0^2)\prod_1^\infty \frac{(1-i\xi/\tau_n)\,e^{\frac{ik\xi}{n\pi}}\,(1-\xi^2/\sigma_n^2)}{(1-i\xi/\kappa_{1,n})\,e^{\frac{ik\xi}{n\pi}}\,(1-\xi^2/j^2_{1,n})} \qquad (132)$$

and

$$Q_-(\xi) = \prod_1^\infty \frac{(1+i\xi/\kappa_{1,n})\,e^{-ik\xi/n\pi}}{(1+i\xi/\tau_n)\,e^{-ik\xi/n\pi}} \qquad (133)$$

This factorisation makes $Q_-(0) = 1$. Similarly the factorisation may be performed for the other cases $m\lambda < 1$ and $m\lambda = 1$. In all cases the similarity with the factorisation in the corresponding two-dimensional problem is immediately evident.

The factorisation has been performed so that the functions (132) and (133) are analytic and non-zero in the upper half plane $\mathcal{J}m\,\xi > 0$ and the overlapping half plane $\mathcal{J}m\,\xi < \kappa_{1,0}$, respectively. The path of integration C in (125), (126) and (127) is taken in this strip. The contour of integration may be closed by an infinite semicircle taken in either the upper or lower half plane. In the former case the integrand of (125) has no poles in this region and, by using the asymptotic forms for the Bessel functions, it may be shown that $\psi(t,y)$ is zero in the region upstream of the leading characteristic (or Mach cone in this axially symmetrical problem) $t + y/h - 1 = 0$.

The formal values of $\psi(t,y)$ may now be obtained from

(125) and (126). The results are very similar to those obtained for the two-dimensional case (equations (65) to (68)) the trigonometrical functions being replaced by Bessel functions. For example, when the semicircle is taken in the lower half plane the contour encloses the poles $\zeta = 0$, $\zeta = \pm\sigma_0$, $\zeta = \pm\sigma_n$ and $\zeta = -i\tau_n$ of the integrand of (125). Hence, in the supersonic region and downstream of the Mach cone from the orifice the potential $\phi(x,y)$ is given by

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = \text{Constant} + \frac{\lambda x}{\beta_1 h(m\lambda - 1)}$$

$$- \sum_{r=0}^{\infty} \frac{J_0(\sigma_r y/h)\ \left[A_r \sin(x\sigma_r/\beta_1 h) + B_r \cos(x\sigma_r/\beta_1 h)\right]}{\sigma_r^2\, J_1(\sigma_r)\quad Q'(\sigma_r)}$$

$$- \sum_{r=1}^{\infty} \frac{e^{-\tau_r x/\beta_1 h}\ I_0(\tau_r y/h)\quad \left[Q-(-i\tau_r)\right]^{-1}}{\tau_r^2\quad I_1(\tau_r)\quad Q'(-i\tau_r)} \qquad (134)$$

where A_r and B_r are given by (64) and $Q(\zeta)$ by (128). Likewise, in the subsonic region the potential $\phi(x,y)$ is given in a form corresponding to (67) and (68) of the two-dimensional problem.

It is observed that far downstream of the jet exis when $x \to \infty$, the second summation in (134) tends to zero and only the oscillatory terms remain. When, in addition, the outer stream width H tends to infinity so that k tends to infinity, the Bessel functions $K_1(k_1\sigma)$ and $I_1(k_1\sigma)$ tend to zero and infinity respectively. The roots σ_n of (128) approximate rapidly to the roots of $\dfrac{K_0(\beta\sigma)}{K_1(\beta\sigma)} = m\,\dfrac{J_0(\sigma)}{J_1(\sigma)}$

which is just the eigenvalue equation quoted by Pai (20 Pg.55). The solution obtained by Pai is the asymptotic ($x \to \infty$) form of (134) when the outer stream is of infinite width.

Although the solutions obtained for the axially symmetrical problem have the same form as those of the two-dimensional problem there are two major differences. The first difference is in the ultimate width of the jet boundary and the second concerns the singularities which appear in the supersonic flow.

Consider the ultimate width of the jet boundary given by $y = h[1 + F(x)]$. In the two-dimensional case this leads to (70), i.e.

$$\frac{F(x)}{\varepsilon \beta_1 \beta_2} \sim \frac{k}{(mk - 1)} + \text{terms in } \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\sigma_r x}{\beta_1 h}\right),$$

From (127) the corresponding solution in the axially symmetrical case is found by evaluating the residue with the greatest imaginary part, i.e.

$$\frac{F(x)}{\varepsilon \beta_1 \beta_2} \sim \frac{\lambda}{2(m\lambda - 1)} + \text{terms in } \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\sigma_r x}{\beta_1 h}\right)$$

Since $\lambda > k$ it follows that $\frac{\lambda}{m\lambda - 1} < \frac{k}{mk - 1}$ for $m\lambda > 1$ and that the jet boundary, in the axially symmetrical case has a mean displacement which is less than half of the corresponding displacement in the two-dimensional case. When the outer stream is infinitely wide the ultimate mean displacement is $\frac{\varepsilon\beta_1\beta_2}{2m}$ which is exactly one half of the value obtained in the two-dimensional case.

To examine the second difference consider the transform $\bar{\psi}(\xi, y)$ in the supersonic region, i.e.

$$\bar{\psi}(\xi, y) = \frac{J_0(\xi y/h)}{\xi^2 J_1(\xi)\, Q_+(\xi)}$$

and let $|\xi|$ be large. The Bessel functions are replaced by their asymptotic forms and then expanded in powers of exponentials. From the Appendix 4, $Q_+(\xi)$ is $O(\xi^q)$ for large $|\xi|$ and hence

$$\bar{\psi}(\xi, y) \sim \frac{A.\ e^{i\xi(1-y/h)}}{(hy)^{1/2}\ (i\xi)^{q+2}}$$

which shows that $\psi(t, y)$ has a discontinuity across the leading Mach cone $t + y/h - 1 = 0$ which tends to infinity as $y^{-\frac{1}{2}}$ as y tends to zero. The potential and the perturbation velocities thus have an infinity at the point on the axis corresponding to the vertex of the Mach cone. This type of singularity is of the same character as that found by Ward[30] and Meyer[32] and is essentially due to the assumption of a linearised theory. Meyer calls this effect the radial focussing effect of axially symmetrical flow because the disturbances which originate inside the jet at the edges of the orifice increase in magnitude on the downstream Mach cone through the disturbances as the axis of the jet is approached. As Ward remarks such singularities cannot be accepted but they do give an indication of a possible normal disc shock-wave which can have no representation in the linear theory.

Finally, it may be shown that the initial slope of the jet boundary is the same as in the two-dimensional case. This is in consequence of the growth of the function $Q_+(\zeta)$ being the same in both problems.

Conclusions

It has been shown possible to solve the problem of an axially symmetrical supersonic jet embedded in an axially symmetrical subsonic stream by applying the Wiener-Hopf technique. Thesolutions are derived in an infite series form. The formal expressions for the perturbation pontentials are shown, in most respects, to be identical in form with those obtained in the two-dimensional problem of Chapter I. This is an immediate consequence of the similarities between the trigonometric and Bessel functions which appear in the respective problems. It is shown that the jet boundary has a fluctuation about a mean displacement and that the jet has an almost periodic structure. It is further shown that the far downstream solution in the case when the outer stream is of infinite width provides the solution found by Pai. Although the forms of the solutions in the axially symmetrical and the two-dimensional cases are similar there are certain differences. One major difference is found to be that the jet displacement in the problem with axial symmetry is less than or equal to one half of the displacement in the two-dimensional problem; the equality holds true when the outer stream is of infinite width. Another major difference arises in the presence of certain singularities in the supersonic flow in the axially symmetrical case. It is shown that across the leading Mach cone from the orifice edge there is a discontinuity in the velocity of order $y^{-\frac{1}{2}}$ giving an infinite singularity at the vertex of the Mach cone. This singularity is an example of the radial focussing effect and it seems probable that it is the nearest representation to a shock wave that the linearised theory can produce.

Despite the fact that the singularities arising cannot be accepted the linearised theory gives a solution which provides an indication, if not a first approximation, of the jet flow. The correctnass of the solutions presented under this theory can be estimated when a general solution to the non-linear problem has been found.

CHAPTER VI

The uniform axially symmetrical supersonic jet in an axially symmetrical supersonic stream

This problem with an infinite outer stream has been discussed in some detail by Pack[(15)] who employed Laplace transform methods. In his discussion it was necessary to deal with certain combinations of Bessel functions which led him to a logarithmic branch point in the inverse integrals involved. If the outer stream is restricted to a finite radius then it is found that the effect of the reflections, at the outer walls, is to introduce additional Bessel functions but that the combinations so formed have no branch points. The inverse integrals may then be evaluated using only the residue theory. The velocity potentials and the jet displacement can then be found in exact forms. Although the solutions are complicated by the presence of the reflected waves from the outer wall they can be readily written down without approximation. The effect of the reflections disappears as the outer walls recede towards infinity and for large values of the outer stream radius the jet behaviour will not be far different from that of a jet in an infinite stream.

The problem may be formulated exactly as in the preceding chapter and presented as a Wiener-Hopf type problem. On satisfying all the boundary conditions the following results are derived:

$$\frac{\bar{\vartheta}_+(\alpha, \ell)\, K(\alpha)}{\beta_2} = \frac{\varepsilon}{\alpha\sqrt{2\pi}} + H_-(\alpha, \ell) \;,$$

where $$K(\alpha) = \frac{Y_1(\alpha\beta_2 H)\, J_0(\alpha\beta_2 h) \;-\; J_1(\alpha\beta_2 H)\, Y_0(\alpha\beta_2 h)}{J_1(\alpha\beta_2 H)\, Y_1(\alpha\beta_2 h) \;-\; Y_1(\alpha\beta_2 H)\, J_1(\alpha\beta_2 h)} + \frac{m\, J_0(\alpha\beta_1 h)}{J_1(\alpha\beta_1 h)}$$

and where $H_-(\alpha, h)$ is a function which is of the form

$$c\,[\,\bar{\Phi}_-(\alpha, h+0) - e^2\,\bar{\Phi}_-(\alpha, h-0)]$$

Since $H_-(\alpha, h)$ is zero from the nature of the supersonic flow it follows that the Wiener-Hopf type equation reduces to

$$\bar{V}_+(\xi, h)\; Q(\xi) \;-\; 1/\xi \;=\; 0 \qquad (136)$$

where

$$Q(\xi) = \frac{Y_1(k_1\xi)\, J_0(\beta\xi) \;-\; J_1(k_1\xi)\, Y_0(\beta\xi)}{J_1(k_1\xi)\, Y_1(\beta\xi) \;-\; Y_1(k_1\xi)\, J_1(\beta\xi)} + \frac{m\, J_0(\xi)}{J_1(\xi)} \qquad (137)$$

The kernel $Q(\xi)$ of (137) behaves in much the same way as the corresponding function $(\cot k\xi + m \cot \xi)$ of the two-dimensional problem. It may be seen that $Q(\xi)$ has no imaginary zeros and it may readily be shown to have no complex zeros. It follows, by the method of Appendix 3, that all its zeros are real and that the left side of (136) is analytic in the upper half plane $\mathcal{I}m\,\xi > 0$. No product factorisation is necessary and, by using the equation (44), the perturbation potentials are obtained from

$$\psi(t, y) = -\frac{1}{\sqrt{2\pi}} \int_c \frac{e^{-i\xi t}\; J_0(\xi y/h)\; d\xi}{\xi^2\; J_1(\xi)\;\; Q(\xi)} \qquad (138)$$

$$0 \leq y \leq h-0,$$

and $$\psi(t,y) = -\frac{1}{\beta\sqrt{2\pi}} \int_C \frac{e^{-i\xi t}\left[Y_1(k_1\xi)\, J_0(\beta\xi y/h) - J_1(k_1\xi)\, Y_0(\beta\xi y/h)\right] d\xi}{\xi^2 \left[J_1(k_1\xi)\, Y_1(\beta\xi) - Y_1(k_1\xi)\, J_1(\beta\xi)\right] Q(\xi)} \tag{139}$$

$$h+0 \leq y \leq H-0,$$

where $Q(\xi)$ is given by (137) and C is a straight line drawn from $-\infty + i\gamma$ to $+\infty + i\gamma$ and γ is a small positive constant. Also the fluctuations of the jet boundary are given by

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_C \frac{e^{-i\xi t}\, d\xi}{\xi^2 Q(\xi)} \tag{140}$$

The logarithmic terms appearing in the Bessel functions Y_0 and Y_1 of (137) cancel and, in consequence, the integrals (138), (139) and (140) may be evaluated by residue theory. The poles of the integrands are at $\xi = 0$ and $\xi = \pm\sigma_n$ where σ_n are the roots of $Q(\xi) = 0$

The function $\psi(t,y)$ and hence the perturbation potentials $\phi(x,y)$ in the various regions of the flow may easily be obtained. The results correspond exactly, in form, with the results in the corresponding two-dimensional problem. Consider the solution of (140) in detail.

Closure of the contour by an infinite semicircle in the upper half plane gives $F(t) = 0$ as it should. Closure of the contour in the lower half plane gives the result

$$\frac{f(x)}{\varepsilon\beta_1\beta_2} = \frac{\lambda}{2(m\lambda+1)} + 2\sum_{r}^{\infty} \frac{\cos(\sigma_r x/\beta_1 h)}{\sigma_r^2\, Q'(\sigma_r)}$$

after use is made of (44), and the fact that $Q'(\xi)$ is an even function.

The jet boundary therefore expands (for $\varepsilon > 0$) to a mean width given by $y = h\left[1 + \frac{\varepsilon \beta_1 \beta_2 \lambda}{2(m\lambda+1)}\right]$ and fluctuates about this mean. For an infinite outer stream the displacement is given by $f(x) = \frac{\varepsilon \beta_1 \beta_2}{m}$ which is exactly one half the value found in the corresponding two-dimensional case. This confirms the findings of Pack. The behaviour of the jet boundary at the jet orifice may be examined by taking the asymptotic form of $\bar{F}(\xi)$ and transforming term by term. From (137) the asymptotic form of $Q(\xi)$ is $\cot k\xi + m \cot(\xi - \pi/4)$ and it is possible to expand $\bar{F}(\xi)$ in a series of powers of $e^{i\xi}$ and $e^{ik\xi}$. The fluctuations in the boundary correspond exactly to those of the two-dimensional case. In particular

$$\bar{F}(\xi) = -\frac{1}{i\xi^2(1+m)}\left[1 + \text{exponential terms}\right]$$

and so,

$$F(t) = \frac{\sqrt{2\pi}}{i(1+m)}\left[t\, H(t) + \cdots\right]$$

where $H(t) = 1, \text{ if } t > 0$
$\qquad\quad = 0, \text{ if } t < 0.$

With (44), this leads to

$$f(x) = \frac{\varepsilon \beta_1 \beta_2 x}{(1+m)\beta_1 h} + \cdots$$

The initial gradient of the jet is then $\frac{\varepsilon \beta_2}{h(1+m)}$ and the jet has a small initial increase. This result is the same as in the two-dimensional case and is in agreement with the result obtained by Pack.

The discontinuities in the flow other than those in the boundary may be examined by the methods employed by Ward[30]. In the transforms for $\bar{\Psi}(\xi, y)$ of (138) and (139) the Bessel functions are replaced by their asymptotic forms and only first terms retained, then

$$\frac{\partial \bar{\Psi}}{\partial y} \sim \frac{\sin(\xi y/h - \pi/4)}{(hy)^{1/2}\ \xi \sin(\xi - \frac{\pi}{4})\left[\cot k\xi + m \cot(\xi - \frac{\pi}{4})\right]} \sim \frac{e^{i(1-y/h)\xi}}{(hy)^{1/2}(1+m)(-i\xi)}$$

$$0 \leq y \leq h-0,$$

and

$$\frac{\partial \bar{\Psi}}{\partial y} \sim \frac{\sin(k_1 - \frac{\beta y}{h})\xi}{-(hy)^{1/2}(\sin k\xi)\left[\cot k\xi + m \cot(\xi - \frac{\pi}{4})\right]} \sim \frac{e^{i(k-k_1+\frac{\beta y}{h})\xi}}{-(hy)^{1/2}(1+m)(-i\xi)}$$

$$h+0 \leq y \leq H-0.$$

These results show that, in both the jet and the stream, the perturbation velocities in the radial directions contain a term in $y^{-1/2}$ and there is a discontinuity in velocity across the line $x/\beta_1 + y - h = 0$ in the jet and across the line $x/\beta_2 - y + h = 0$ in the stream. These lines correspond to the leading characteristics from the jet orifice in the axially symmetrical flow. In particular there is an infinity on the jet axis where the Mach cone from the orifice edge has its vertex. This indicates the presence of focussing effects such as the start of a shock wave or expansion wave.

Conclusion

The problem of an axially symmetrical supersonic jet embedded in an axially symmetrical supersonic stream of finite width has been solved by using Fourier transform theory. Although presented here as an example of the Wiener-Hopf technique there is strictly no necessity to introduce this technique. The inverse transform integrals are readily evaluated using only the residue theory. This is a consequence of the introduction of the outer walls. For an infinite outer stream Pack has shown that the integrals contain a logarithmic branch point. It is shown that the form of the solution in the axially symmetrical case is the same as in the two-dimensional case. There are, however, fundamental differences in the flows. In the problem with axial symmetry the jet boundary converges rapidly to an asymptotic radius of $\left[1 + \frac{\varepsilon\beta_1\beta_2\lambda}{2(m\lambda+1)}\right]$ times the nozzle radius. In the case of an infinite outer stream the value agrees with that found by Pack: the mean displacement being exactly one half of the value found in the corresponding two-dimensional case.

It is possible to expand the various inverse transforms in a series of powers of exponential functions, transform term by term, and derive the pattern of the characteristics. Although this pattern appears to agree with that of the two-dimensional case it is doubtful if the results are even approximately true. As Meyer has remarked the linear theory cannot permit conclusions to be drawn regarding the flow pattern, in axially symmetrical problems, in the neighbourhood and downstream of the reflected Mach lines. In the present problem, it is shown that there are discontinuities in the radial perturbation velocities across the leading Mach cones from the orifice edge which behave like $y^{-\frac{1}{2}}$. Inside the jet, the discontinuity is infinite at the vertex of the Mach cone on the jet axis, an example of the focussing effect of axial flow. Such effects are not found in the two-dimensional case where a Mach line belongs either to a region of uniform flow or to a simple wave.

CHAPTER VII

The uniform axially symmetrical subsonic jet in a uniform axially symmetrical supersonic stream

This problem will not be discussed in any detail since it is basically similar to the problem of Chapter V. It is sufficient to note that in this case the equations (125) to (128) of the former Chapter are replaced by

$$\psi(t,y) = \frac{1}{\sqrt{2\pi}} \int_c \frac{e^{-i\xi t}\, I_0(\xi y/h)\cdot Q_-(0)\, d\xi}{\xi^2\, I_1(\xi)\, Q_+(\xi)}, \tag{141}$$

$$0 \leq y \leq h-0;$$

$$\psi(t,y) = \frac{1}{\sqrt{2\pi}} \int_c \frac{e^{-i\xi t}\left[Y_1(k_1\xi) J_0(\beta\xi y/h) - J_1(k_1\xi) Y_0(\beta\xi y/h)\right] Q_-(0)\, d\xi}{\xi^2 \left[J_1(k_1\xi) Y_1(\beta\xi) - Y_1(k_1\xi) J_1(\beta\xi)\right] Q_+(\xi)} \tag{142}$$

$$h+0 \leq y \leq H-0,$$

and

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_c \frac{e^{-i\xi t}\, Q_-(0)\, d\xi}{\xi^2\, Q_+(\xi)}, \tag{143}$$

where

$$Q(\xi) = \frac{Q_+(\xi)}{Q_-(\xi)} = \frac{J_1(k_1\xi) Y_0(\beta\xi) - Y_1(k_1\xi) J_0(\beta\xi)}{Y_1(k_1\xi) J_1(\beta\xi) - J_1(k_1\xi) Y_1(\beta\xi)} - m\frac{I_0(\xi)}{I_1(\xi)}. \tag{144}$$

The first quotient in $Q(\xi)$ has the behaviour of a cotangent (for real ξ) and is asymptotic to $\cot k\xi$, the second quotient has an infinity at $\xi = 0$ and steadily decreases to the asymptotic value of unity. It has the behaviour (for real ξ) of $\coth \xi$. The roots $\xi = \pm \sigma_n$ and $\xi = \pm i \tau_n$ all lie on the real and imaginary axes and are such that for large n

$$k \sigma_n \sim n\pi + \phi ,$$
$$\tau_n \sim n\pi + \theta + \frac{\pi}{4} .$$

The solution to the problem may be obtained in the usual way after factorisation of $Q(\xi)$. It may be observed that the solution for the case $m\lambda > 1$ will be very similar to the solution of the problem of Chapter V for the case $m\lambda < 1$. This similarity between the two problems has already been noticed in the corresponding two-dimensional problems of Chapters I and II. In particular, the ultimate mean jet width will be given by

$$\frac{f(x)}{\varepsilon \beta_1 \beta_2} = - \frac{\lambda}{2(m\lambda - 1)}$$

which shows that in the over-pressure case there is a slight diminution in the jet width when $m\lambda > 1$.

CHAPTER VIII

The uniform subsonic axially symmetrical jet in a uniform subsonic axially symmetrical stream

The formulation of this problem is as given in Chapter V. If β_1 be replaced by $i\beta_1$ the following equations may be obtained

$$\psi(t,y) = \frac{1}{\sqrt{2\pi}} \int_c \frac{e^{-i\xi t} \; I_0(\xi y/h) \; \bar{V}_+(\xi, h) \, d\xi}{\xi \quad I_1(\xi)} \qquad (145)$$

$$0 \leq y \leq h-0 \; ;$$

$$\psi(t,y) = -\frac{1}{\beta\sqrt{2\pi}} \int_c \frac{e^{-i\xi t} \left[I_1(k_1\xi) K_0(\beta\xi y/h) + K_1(k_1\xi) I_0(\beta\xi y/h) \right] \bar{V}_+(\xi, h) \, d\xi}{\xi \left[I_1(k_1\xi) K_1(\beta\xi) - K_1(k_1\xi) I_1(\beta\xi) \right]} \qquad (146)$$

$$h+0 \leq y \leq H-0 \; ;$$

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_c \frac{e^{-i\xi t} \quad \bar{V}_+(\xi, h) \, d\xi}{\xi} \; ; \qquad (147)$$

where

$$\bar{V}_+(\xi, h) = - \frac{Q_-(0)}{\xi \, Q_+(\xi)}$$

and

$$Q(\xi) = \frac{Q_+(\xi)}{Q_-(\xi)} = \frac{I_1(k_1\xi) K_0(\beta\xi) + K_1(k_1\xi) I_0(\beta\xi)}{I_1(k_1\xi) K_1(\beta\xi) - K_1(k_1\xi) I_1(\beta\xi)} + m \frac{I_0(\xi)}{I_1(\xi)} . \qquad (148)$$

It may be shown by an argument similar to that given in Appendix 3 that $Q(\xi)$ has no complex zeros. An examination of $Q(\xi)$, with ξ real, shows that both quotients are positive. The zeros of $Q(\xi)$ are thus all imaginary and given by $\xi = \pm i\tau_n$ where τ_n are the zeros of the function (137) which behaves like $\cot k\xi + m\cot(\xi - \pi/4)$ for large $|\xi|$. The poles of $Q(\xi)$ are all simple and imaginary and the factorisation of $Q(\xi)$ leads to

$$Q_+(\xi) = \frac{2(1+m\lambda)}{\lambda\xi} e^{\chi(\xi)} \prod_1^\infty \frac{\left(1 - \frac{i\xi}{\tau_n}\right) e^{i\xi/\tau_n}}{\left(1 - i\xi/j_{1,n}\right) e^{i\xi/n\pi} \left(1 - i\xi/K_{1,n}\right) e^{ik\xi/n\pi}}$$

and

$$Q_-(\xi) = e^{\chi(\xi)} \prod_1^\infty \frac{\left(1 + i\xi/j_{1,n}\right) e^{-i\xi/n\pi} \left(1 + i\xi/K_{1,n}\right) e^{-ik\xi/n\pi}}{\left(1 + i\xi/\tau_n\right) e^{-i\xi/\tau_n}}$$

where $j_{1,n}$ and $K_{1,n}$ are the non-vanishing zeros of $J_1(z)$ and $J_1(k_1 z) Y_1(\beta z) - Y_1(k_1 z) J_1(\beta z)$ respectively, both arranged in ascending order of magnitude. The functions must be chosen so that $Q_+(\xi)$ and $Q_-(\xi)$ are of algebraical growth order as $|\xi| \to \infty$. From a consideration of the Gamma functions which replace the product functions of $Q_\pm(\xi)$ for large $|\xi|$, it may be shown that $\chi(\xi)$ is linear in ξ and that $\chi(0) = 0$. This makes $Q_-(0) = 1$.

It may be argued on physical grounds that the integral function $E'(\xi)$, which appears on application of the Wiener-Hopf technique, is zero, and the calculations leading to the perturbation potentials and the jet displacement follow the usual procedure. Thus

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = \frac{\lambda\left[x - i\beta_1 h\, Q_-'(0)\right]}{\beta_1 h (1+m\lambda)} + \sum_1^{\infty} \frac{e^{-x\gamma_n/\beta_1 h}\, J_0(\gamma_n y/h)\, \left[Q_-(-i\gamma_n)\right]^{-1}}{\gamma_n^2\, J_1(\gamma_n)\, Q'(-i\gamma_n)}, \tag{149}$$

in the region $x>0,\ 0 \le y \le h-0$;

$$\frac{\phi(x,y)}{\varepsilon h \beta_2} = \sum_1^{\infty} \frac{e^{x j_{1,n}/\beta_1 h}\, J_0(y j_{1,n}/h)\, \bar{V}_+(i j_{1,n}; h)}{j_{1,n}\, J_1(j_{1,n})}, \tag{150}$$

in the region $x<0,\ 0 \le y \le h-0$;

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} = -\left[\frac{x - i\beta_1 h\, Q_-'(0)}{\beta_1 h (1+m\lambda)}\right] - \sum_1^{\infty} \frac{e^{-\gamma_n x/\beta_1 h}\left[J_1(k_1\gamma_n) Y_0(\beta\gamma_n y/h) - Y_1(k_1\gamma_n) J_0(\beta\gamma_n y/h)\right]\left[Q_-(-i\gamma_n)\right]^{-1}}{\gamma_n^2 \left[Y_1(k_1\gamma_n) J_1(\beta\gamma_n) - J_1(k_1\gamma_n) Y_1(\beta\gamma_n)\right] Q'(-i\gamma_n)} \tag{151}$$

in the region $x>0,\ h+0 \le y \le H-0$; and

$$\frac{\phi(x,y)}{\varepsilon h \beta_1} = -\sum_1^{\infty} \frac{e^{x K_{1,n}/\beta_1 h}\, P(iK_{1,n})\, \bar{V}_+(iK_{1,n}; h)}{K_{1,n}} \tag{152}$$

in the region $x<0,\ h+0 \le y \le H-0$;

where, in (152),

$$P(\zeta) = \frac{I_1(k_1\zeta) K_0(\beta\zeta) + K_1(k_1\zeta) I_0(\beta\zeta)}{\frac{d}{d\zeta}\left(I_1(k_1\zeta) K_1(\beta\zeta) - K_1(k_1\zeta) I_1(\beta\zeta)\right)} \tag{153}$$

It may be remarked that the expression (153) may be considerably reduced by the introduction of the cylinder functions adopted by Jaeger[31] and defined by

$$D(x,y) = I_0(x)\,K_0(y) - K_0(x)\,I_0(y)$$

and

$$C(x,y) = J_0(x)\,Y_0(y) - Y_0(x)\,K_0(y)$$

with
$$D_{r,s}(x,y) = \frac{\partial^{r+s}}{\partial x^r\,\partial y^s} D(x,y).$$

For the present purpose it seems sufficient to observe that the results (149) to (152) closely resemble the results obtained for the corresponding two-dimensional problem. The disturbances due to the interaction of the jet and stream are transmitted upstream as well as down, a result as anticipated by the nature of the subsonic flow. In particular there are no periodic, or almost periodic, terms in the x variable present in the solution and the disturbances all decay, upstream and downstream, exponentially with the distance from the jet exit.

The disturbances in the jet boundary are examined by evaluating the function F(t) of (147). Now

$$\bar{F}(\zeta) = -\frac{1}{\zeta^2 Q_+(\zeta)}$$

and $Q_+(\zeta)$ is given above. It follows that

$$F(t) = -\frac{1}{\sqrt{2\pi}}\int_C \frac{e^{-i\zeta t}\,d\zeta}{\zeta^2 Q_+(\zeta)\,Q_-(\zeta)}$$

and the integrand has simple poles at $\zeta = 0$ and $\zeta = -i\tau_n$.
Thus

$$F(t) = i\sqrt{2\pi}\left[\frac{\lambda}{2(1+m\lambda)} - \sum_{1}^{\infty}\frac{e^{-\tau_n t}\,[Q_-(-i\tau_n)]^{-1}}{\tau_n^2\,[Q'(-i\tau_n)]}\right]$$

which leads to

$$\frac{f(x)}{\varepsilon\beta_1\beta_2} = -\frac{\lambda}{2(1+m\lambda)} + \sum_{1}^{\infty}\frac{e^{-\tau_n x/\beta_1 h}\,[Q_-(-i\tau_n)]^{-1}}{\tau_n^2\,[Q'(-i\tau_n)]} \qquad (154)$$

for $x > 0$. From the location of the poles it is seen that $f(x) = 0$ for $x < 0$.

As in the two-dimensional case the result (154) shows that the jet quickly converges to complete uniformity. The ultimate width in the axially symmetrical problem is given by

$$\frac{f(x)}{\varepsilon\beta_1\beta_2} \sim -\frac{\lambda}{2(1+m\lambda)}$$

where $\lambda = \beta\left(\frac{H^2}{h^2} - 1\right)$.

The corresponding result in the two-dimensional problem is

$$\frac{f(x)}{\varepsilon\beta_1\beta_2} \sim -\frac{k}{(1+mk)}$$

where $k = \beta\left(\frac{H}{h} - 1\right)$.

As the outer walls recede and the stream radius becomes infinite the displacement tends to $\frac{\varepsilon\beta_1\beta_2}{2m}$ which is exactly one half of the value given in the two-dimensional case. In the overpressure case, ($\varepsilon > 0$) the jet boundary steadily diminishes to its ultimate width.

<u>Conclusion</u>

The solution of a uniform axially symmetrical subsonic jet embedded in an axially symmetrical subsonic stream has been found in a closed form by an application of the Wiener-Hopf technique. The combinations of Bessel functions which appear in the problem are capable of infinite product factorisation in the same way as in the two-dimensional problem. It is shown that the character of the solutions, in the corresponding problems, is similar and that the perturbation terms decay exponentially with the distance from the jet exit. The ultimate width of the jet boundary differs from the value in the two-dimensional problem; in particular, when the outer stream is of infinite radius the displacement is exactly one half of the value found in the two-dimensional case.

APPENDICES

Appendix 1

It is required to show that the roots of $\tan z - m \tanh kz = 0$ where m and k are positive, lie only on the axes. When $z = x + iy$ the equation $z = m \tanh kz$ may be written

$$\frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y} = m \frac{\sinh 2kx + i \sin 2ky}{\cosh 2kx + \cos 2ky}$$

It follows that the two equations

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} = m \frac{\sinh 2kx}{\cosh 2kx + \cos 2ky}$$

and

$$\frac{\sinh 2y}{\cos 2x + \cosh 2y} = m \frac{\sin 2ky}{\cosh 2kx + \cos 2ky}$$

are true simultaneously.

The conclusion is that if there are roots for which neither x nor y is zero, then for some $x \neq 0$, $y \neq 0$:-

$$\sin 2x \sin 2ky = \sinh 2y \sinh 2kx$$

Whatever the values of x, y ; positive or negative, attention may be restricted to the first quadrant only.

Now $\sin \theta < \theta$ and $\sinh \phi > \phi$ for all θ and ϕ and so the left side of 1.1 is less than $4kxy$ whilst the right side is greater than $4kxy$. Hence 1.1 cannot be true except for $x = 0$ or $y = 0$.

The roots of $\tan z - m \tanh kz = 0$ therefore lie only along the two axes of the z-plane. The roots may be evaluated by graphical or numerical methods.

Appendix 2

To determine the integral function $E(\alpha)$ of equations (37) and (38) it is necessary to investigate the order of growth at infinity (in the appropriate half plane) of the various functions which appear in the two sides of (36). From (36), with $K_-(0) = 1$,

$$\frac{\bar{\vartheta}_+(\alpha, h)\; K_+(\alpha)}{\beta_2} + \frac{\varepsilon}{\alpha\sqrt{2\pi}} = E(\alpha) \quad ; \quad \mathcal{I}m\,\alpha > 0,$$

$$-\frac{\varepsilon\left[K_-(\alpha) - 1\right]}{\alpha\sqrt{2\pi}} + \bar{k}_-(\alpha, h)\; K_-(\alpha) = E(\alpha) \quad ; \quad \mathcal{I}m\,\alpha < \rho.$$

An application of the transformations (43) leads to

$$\bar{V}_+(\xi, h)\; Q_+(\xi) + \xi^{-1} = E'(\xi) \quad , \quad \mathcal{I}m\,\xi > 0, \tag{2.1}$$

$$-\frac{\left[Q_-(\xi) - 1\right]}{\xi} + \bar{k}_-(\xi)\, Q_-(\xi) = E'(\xi) \quad , \quad \mathcal{I}m\,\xi < \rho', \tag{2.2}$$

where $E'(\xi) = C\,E(\alpha)$ and C is a constant independent of α. The functions $Q_+(\xi)$ and $Q_-(\xi)$ are given, in infinite product form, by (51) and (52) for $mk > 1$; by (54) and (55) for $mk < 1$ and by (57) and (58) for $mk = 1$. To investigate the growth of $Q_+(\xi)$ it is necessary to examine the growth order of the product function

$$L(\xi) = \prod_{1}^{\infty} \frac{(1 - i\xi/\tau_n)\, e^{ik\xi/n\pi}\, (1 - \xi^2/\sigma_n^2)}{(1 - ik\xi/n\pi)\, e^{ik\xi/n\pi}\, (1 - \xi^2/n^2\pi^2)} \tag{2.3}$$

as $|\zeta| \to \infty$, in the region $\mathcal{I}_m \zeta > 0$.

It has been noted that for large n

$$\sigma_n \sim n\pi + \tan^{-1} m ,$$

$$k\tau_n \sim n\pi + \tan^{-1}\left(\tfrac{1}{m}\right).$$

Write

$$\left.\begin{aligned} \sigma_n &\sim n\pi + \theta = \pi(n+p) \\ \text{and} \quad k\tau_n &\sim n\pi + \phi = \pi(n+q) , \end{aligned}\right\} \qquad (2\cdot4)$$

where p and q are each less than 1/2 and p + q = 1/2.

Follow Noble (19 Pg.128) and consider the asymptotic behaviour of the following expression:

$$H(z) = \prod_1^\infty (1 + z/z_n)\, e^{-\frac{z}{n}} \qquad (2\cdot5)$$

when $z_n = n + t + o(1/n)$ for large values of n. $\qquad (2\cdot6)$

Compare the behaviour of H(z) as $z \to \infty$ with that of

$$J(z) = \prod_1^\infty \left(1 + \frac{z}{n+t}\right) e^{-\frac{z}{n}}$$

$$= \frac{e^{-\gamma z}\, \Gamma(1+t)}{\Gamma(1+t+z)} \qquad (2\cdot7)$$

where γ is the Euler-Mascheroni constant and $\Gamma(z)$ is the Gamma function (21). Then

$$\frac{H(z)}{J(z)} = L(z) \prod_1^\infty \frac{(n+t)}{z_n} , \qquad (2\cdot8)$$

where $$L(z) = \prod_1^\infty \frac{z_n + z}{n+t+z} = \prod_1^\infty \left[1 + \frac{z_n - n - t}{z + n + t}\right] .$$

With the use of 2.6 it may readily be shown that

$$\left| \frac{z_n - n - t}{z + t + n} \right| < \frac{C}{n^2}$$

where C is a constant independent of z, for all z such that $\Im m\, z > \eta > 0$. The function L(z) is therefore uniformly convergent and

$$\lim_{z \to \infty} L(z) = \prod_1^{\infty} \lim_{z \to \infty} \left[1 + \frac{z_n - n - t}{z + n + t} \right] = 1 .$$

Thus, from 2.8, for large z

$$H(z) \sim J(z) . \tag{2.9}$$

Return to 2.3 and write it in the form

$$L(\zeta) = \left(\frac{\prod_1^{\infty} \left(1 - \frac{ik\zeta/\pi}{k\gamma_n/\pi}\right) e^{\frac{ik\zeta/\pi}{n}}}{\prod_1^{\infty} \left(1 - \frac{ik\zeta/\pi}{n}\right) e^{\frac{ik\zeta/\pi}{n}}} \right) \left(\frac{\prod_1^{\infty} \left(1 - \frac{\zeta/\pi}{\sigma_n/\pi}\right) e^{\frac{\zeta/\pi}{n}}}{\prod_1^{\infty} \left(1 - \frac{\zeta/\pi}{n}\right) e^{\frac{\zeta/\pi}{n}}} \right) \left(\frac{\prod_1^{\infty} \left(1 + \frac{\zeta/\pi}{\sigma_n/\pi}\right) e^{\frac{\zeta/\pi}{n}}}{\prod_1^{\infty} \left(1 - \frac{\zeta/\pi}{n}\right) e^{\frac{\zeta/\pi}{n}}} \right)$$

From (2.4); (2.9) and (2.7) it follows that, for large z in the region $\Im m\, \zeta > 0+$,

$$L(\zeta) \sim \left(\frac{\Gamma\left(1 - \frac{ik\zeta}{\pi}\right)}{\Gamma\left(1 + q - \frac{ik\zeta}{\pi}\right)} \right) \left(\frac{\Gamma\left(1 - \frac{\zeta}{\pi}\right)}{\Gamma\left(1 + p - \frac{\zeta}{\pi}\right)} \right) \left(\frac{\Gamma\left(1 + \frac{\zeta}{\pi}\right)}{\Gamma\left(1 + p + \frac{\zeta}{\pi}\right)} \right) \tag{2.10}$$

The Stirling expansion formulae for the Gamma function, namely

$$\left.\begin{aligned} &\Gamma(1+z) \sim (2\pi)^{1/2} z^{z+1/2} e^{-z}\left[1 + O\left(\tfrac{1}{z}\right)\right] \\ \text{and} \quad &\Gamma(1+z+a) \sim (2\pi)^{1/2} z^{z+1/2+a} e^{-z}\left(1 + O\left(\tfrac{1}{z}\right)\right) \end{aligned}\right\} \tag{2.11}$$

are applied to (2.10) to give the asymptotic form

$$L(\zeta) \sim \left(\frac{1}{-ik\zeta/\pi}\right)^{q}\left(\frac{1}{-\zeta/\pi}\right)^{p}\left(\frac{1}{\zeta/\pi}\right)^{p}$$

$$= O\left(\frac{1}{\zeta^{q+2p}}\right) = O\left(\zeta^{-(1-q)}\right) \qquad (2.12)$$

By a similar argument it may be shown that the product function which appears in the function $Q_-(\zeta)$ has the order $O(\zeta^{q})$ for large $|\zeta|$ in the lower half plane.

The functions $Q_+(\zeta)$ and $Q_-(\zeta)$ are thus of algebraical growth at infinity and are such that

$$Q_+(\zeta) \sim O(\zeta^{q}); \; Q_-(\zeta) \sim O(\zeta^{q}) \quad \text{when} \quad mk \geq 1,$$

and $$Q_+(\zeta) \sim O(\zeta^{-(1-q)}); \; Q_-(\zeta) \sim O(\zeta^{-(1-q)}) \quad \text{when} \quad mk < 1.$$

In all cases it may be observed that $Q(\zeta) = Q_+(\zeta)/Q_-(\zeta) = O(1)$ within the strip.

Finally, the growth order of the functions $\bar{V}_+(\zeta, h)$ and $G_-(\zeta)$ must be examined. Since $V_+(t,h)$ is assumed to be integrable in $0 < t < L_1$ (where $L_1 > 0$) and $V_+(t,h)$ approaches zero as $t \to 0$, then $V_+(t,h) = O(t^{-r}), [r>0]$ as $t \to 0$. Thus $\bar{V}_+(\zeta, h) = O(\zeta^{-(r+1)})$ as $|\zeta| \to \infty$, $\mathcal{I}m\,\zeta > 0$. Also $G_-(\zeta, h)$ approaches zero as $|\zeta| \to \infty$, $\mathcal{I}m\,\zeta < c'$, in consequence of the Riemann-Lebesgue Lemma. Consider first the cases $mk \geq 1$. In (2.1) the first term on the left side, $\bar{V}_+(\zeta, h)\, Q_+(\zeta)$, is $O[\zeta^{-(r+p+\frac{1}{2})}]$ and the second term is $O(\zeta^{-1})$. Thus $E'(\zeta)$ is zero for large $|\zeta|$, $\mathfrak{Im}\,\zeta > 0$. In (2.2) the first term on the left side, $[Q_-(\zeta) - 1]/\zeta$, is $O[\zeta^{-(1-q)}]$ and the second term is $O(\zeta^q)$. Recall that $q < \frac{1}{2}$, thus $E'(\zeta)$ is, at most, a constant for large $|\zeta|$, $\mathfrak{Im}\,\zeta < e'$. Hence $E'(\zeta)$ is zero for large $|\zeta|$ in the whole ζ -plane, and by Liouville's theorem it is zero everywhere.

Consider the case $mk < 1$. In (2.1) the first term on the left side is $O[\zeta^{-(r+2-q)}]$ and the second term is $O[\zeta^{-1}]$, thus $E'(\zeta)$ is zero for $|\zeta| \to \infty$, $\mathfrak{Im}\,\zeta > 0$. In (2.2), the first term is $O[\zeta^{-(2-q)}]$ and the second is $O[\zeta^{-(1-q)}]$ and $E'(\zeta)$ is zero for $|\zeta| \to \infty$, $\mathfrak{Im}\,\zeta < e'$. Thus, by Liouville's theorem $E'(\zeta)$ is zero everywhere.

Thus for all mk , $E'(\zeta)$, and hence $E(\alpha)$, is zero. It may be noted that $\bar{V}_+(\zeta, a)$ is $O[\zeta^{-(1+q)}]$ if $mk \geq 1$ and is $O[\zeta^{-q}]$ if $mk < 1$.

Appendix 3

It is required to show that the function $K(\alpha)$, or its equivalent $Q(\xi)$, of Chapter V has no complex roots. Now,

$$Q(\xi) = \frac{I_1(k_1\xi)K_0(\beta\xi) + K_1(k_1\xi)I_0(\beta\xi)}{I_1(k_1\xi)K_1(\beta\xi) - K_1(k_1\xi)I_1(\beta\xi)} - m\frac{J_0(\xi)}{J_1(\xi)}$$

where $k_1 = \beta H/h$; β and m are all positive.

Follow the method adopted by Jaeger (26 Pg.324; 28).

Suppose that $\xi = \gamma$ is a zero of $Q(\xi)$ and write

$$U_1 = J_1(\gamma y/h) \;;\quad \text{for } 0 < y < h$$

and

$$U_2 = \left[\frac{I_1(k_1\gamma)K_1(\beta\gamma y/h) - K_1(k_1\gamma)I_1(\beta\gamma y/h)}{I_1(k_1\gamma)K_1(\beta\gamma) - K_1(k_1\gamma)I_1(\beta\gamma)}\right]J_1(\gamma) \quad \text{for } h < y < H.$$

The functions U_1 and U_2 are non-zero solutions of the differential equations

$$\frac{d}{dy}\left[y\frac{dU_1}{dy}\right] + y\left[\frac{\gamma^2}{h^2} - \frac{1}{y^2}\right]U_1 = 0, \quad 0<y<h \qquad (3.1)$$

and

$$\frac{d}{dy}\left[y\frac{dU_2}{dy}\right] - y\left[\frac{\gamma^2\beta^2}{h^2} + \frac{1}{y^2}\right]U_2 = 0, \quad h<y<H \qquad (3.2)$$

respectively.

The boundary conditions on U_1 and U_2 are given by

$$\left.\begin{aligned} U_1 &= 0 && \text{when } y = 0, \\ U_1 &= U_2 && \text{when } y = h, \\ U_2 &= 0 && \text{when } y = H. \end{aligned}\right\} \qquad 3.3$$

The recurrence formulae (29)

$$z\, J_1'(z) + J_1(z) = z\, J_0(z) ,$$

$$z\, I_1'(z) + I_1(z) = z\, I_0(z) ,$$

and $$z\, K_1'(z) + K_1(z) = -z\, K_0(z)$$

may now be used to show that

$$y\,\frac{du_1}{dy} + u_1 = \frac{\gamma y}{h}\, J_0\left(\frac{\gamma y}{h}\right)$$

and

$$y\,\frac{du_2}{dy} + u_2 = -\frac{\beta\gamma y}{h}\left[\frac{I_1(k_1\gamma)\,K_0(\beta\gamma y/h) + K_1(k_1\gamma)\,I_0(\beta\gamma y/h)}{I_1(k_1\gamma)\,K_1(\beta\gamma) - K_1(k_1\gamma)\,I_1(\beta\gamma)}\right] .$$

Thus

$$\left[\left(y\,\frac{du_2}{dy} + u_2\right) + \beta m\left(y\,\frac{du_1}{dy} + u_1\right)\right]_{y=h} = -\beta\gamma\, J_1(\gamma)\, Q(\gamma)$$

$$= 0 \qquad (3{\cdot}4)$$

since $\xi = \gamma$ is a zero of $Q(\xi)$.

Let there now be two different roots $\xi = \gamma$ and $\xi = \delta$ and suppose that U_1 , U_2 are functions, defined above, for $\xi = \gamma$ and that V_1 , V_2 are the corresponding functions when $\xi = \delta$. Then V_1 and V_2 will satisfy (3.1), (3.2), (3.3) and (3.4) on replacing U_1, U_2 by V_1, V_2 and γ by δ . From (3.1), and the corresponding equation in V_1 and δ , the following result is obtained

$$\frac{(\gamma^2 - \delta^2)}{h^2}\int_0^h y\, u_1 v_1\, dy + \int_0^h \left[V_1 \frac{d}{dy}\left(y\,\frac{du_1}{dy}\right) - u_1 \frac{d}{dy}\left(y\,\frac{dV_1}{dy}\right)\right] dy = 0 \qquad (3{\cdot}5)$$

Likewise, from (3.2) and the corresponding equation in V_2 and δ, the following result is obtained

$$\frac{\beta^2(\gamma^2-\delta^2)}{h^2}\int_h^H y\,U_2V_2\,dy + \int_h^H\left[U_2\frac{d}{dy}\left(y\frac{dV_2}{dy}\right) - V_2\frac{d}{dy}\left(y\frac{dU_2}{dy}\right)\right]dy = 0 \qquad (3.6)$$

The second integrals in (3.5) and (3.6) may be integrated by parts and the resultant equations combined in the form

$$\frac{(\gamma^2-\delta^2)}{h^2}\left[\beta m\int_0^h y\,U_1V_1\,dy + \beta^2\int_h^H y\,U_2V_2\,dy\right]$$

$$= \beta m\left[U_1\left(y\frac{dV_1}{dy}\right) - V_1\left(y\frac{dU_1}{dy}\right)\right]_0^h + \left[V_2\left(y\frac{dU_2}{dy}\right) - U_2\left(y\frac{dV_2}{dy}\right)\right]_h^H \qquad (3.7)$$

From (3.3), the right side of (3.7) may be written

$$\left[U_1\left\{\left(y\frac{dV_2}{dy}\right) + \beta m\left(y\frac{dV_1}{dy}\right)\right\} - V_1\left\{\left(y\frac{dU_2}{dy}\right) + \beta m\left(y\frac{dU_1}{dy}\right)\right\}\right]_{y=h}$$

From (3.4), and the associated equation in V_1 and V_2, this may be further reduced to

$$\left[U_1(-V_2-\beta m V_1) - V_1(-U_2-\beta m U_1)\right]_{y=h}$$

$$= \left[V_1U_2 - U_1V_2\right]_{y=h}.$$

This last expression is zero from (3.3) and its associated equation in V_1 and V_2.

Thus

$$\frac{\gamma^2-\delta^2}{h^2}\left[\beta m\int_0^h y\,U_1V_1\,dy + \beta^2\int_h^H y\,U_2V_2\,dy\right] = 0 \qquad (3.8)$$

If $Q(\xi)$ has complex roots then they occur in complex conjugate pairs, i.e. $\xi = \sigma \pm i\tau$. Suppose that γ and δ are such a pair then U_1, V_1 and U_2, V_2 are complex conjugate quantities and the products U_1V_1 and U_2V_2 are both positive. Since β and m are also positive quantities it follows that the square bracket [ ] in (3.8) is positive. Thus

$$(\gamma^2-\delta^2) = 2i\sigma\tau = 0.$$

There are therefore no complex roots; either $\tau = 0$ and the roots are real or $\sigma = 0$ and the roots are imaginary.

The roots may be found graphically or numerically and it is seen that (with the possible exception of the root $\xi = 0$) there are no repeated roots.

Appendix 4

To determine the integral function $E(\alpha)$ which appears in the Wiener-Hopf equation of Chapter V it is necessary to investigate the order of growth at infinity of the functions which appear in equation (36) with the $K(\alpha)$ determined as in Chapter V.

The proof follows the method adopted in Appendix 2. Consider the case $m\lambda > 1$, From (132), it is required to find the growth order of the product function

$$L(\zeta) = \prod_1^\infty \frac{(1 - i\zeta/\gamma_n)\, e^{ik\zeta/n\pi}\,(1 - \zeta^2/\sigma_n^2)}{(1 - i\zeta/K_{1,n})\, e^{ik\zeta/n\pi}\,(1 - \zeta^2/j^2_{1,n})} \qquad (4\cdot1)$$

in which

$$\left.\begin{aligned} k\gamma_n &\sim n\pi + \tan^{-1}(1/m) = \pi(n+q), \\ \sigma_n &\sim n\pi + \tan^{-1} m = \pi(n+p+1/4), \\ j_{1,n} &\sim \pi(n+1/4), \\ K_{1,n} &\sim n\pi/k . \end{aligned}\right\} \qquad (4\cdot2)$$

In Appendix 2, it is shown that

$$\prod_1^\infty (1 + z/z_n)\, e^{-z/n} \sim \frac{e^{-\gamma z}\,\Gamma(1+t)}{\Gamma(1+t+z)} \qquad (4\cdot3)$$

where $z_n = n + t + O(1/n)$ for large n.

Write (4.1) in the form

$$L(\zeta) = \left(\frac{\prod_1^\infty \left(1 - \frac{ik\zeta/\pi}{k\gamma_n/\pi}\right) e^{\frac{ik\zeta/\pi}{n}}}{\prod_1^\infty \left(1 - \frac{ik\zeta/\pi}{k\kappa_{1,n}/\pi}\right) e^{\frac{ik\zeta/\pi}{n}}} \right) \left(\frac{\prod_1^\infty \left(1 - \frac{\zeta/\pi}{\sigma_n/\pi}\right) e^{\frac{\zeta/\pi}{n}}}{\prod_1^\infty \left(1 - \frac{\zeta/\pi}{j_{1,n}/\pi}\right) e^{\frac{\zeta/\pi}{n}}} \right) \left(\frac{\prod_1^\infty \left(1 + \frac{\zeta/\pi}{\sigma_n/\pi}\right) e^{-\frac{\zeta/\pi}{n}}}{\prod_1^\infty \left(1 + \frac{\zeta/\pi}{j_{1,n}/\pi}\right) e^{-\frac{\zeta/\pi}{n}}} \right)$$

If (4.3) be applied to each of the product functions in $L(\zeta)$ then, for large $|\zeta|$ in $\Im m\, \zeta > 0$,

$$L(\zeta) \sim \left(\frac{\Gamma\left(1 - \frac{i\zeta k}{\pi}\right)}{\Gamma\left(1 + q - \frac{i\zeta k}{\pi}\right)} \right) \left(\frac{\Gamma\left(1 + \frac{1}{4} - \frac{\zeta}{\pi}\right)}{\Gamma\left(1 + p + \frac{1}{4} - \frac{\zeta}{\pi}\right)} \right) \left(\frac{\Gamma\left(1 + \frac{1}{4} + \frac{\zeta}{\pi}\right)}{\Gamma\left(1 + p + \frac{1}{4} + \frac{\zeta}{\pi}\right)} \right).$$

The Stirling expansion formulae (2.11) gives the asymptotic form of $L(\zeta)$,

$$L(\zeta) \sim \frac{1}{\left(-\frac{ik\zeta}{\pi}\right)^q} \, \frac{1}{\left(-\frac{\zeta}{\pi}\right)^p} \, \frac{1}{\left(\frac{\zeta}{\pi}\right)^p} = O\left(\frac{1}{\zeta^{1-q}}\right).$$

Thus for $\Re \lambda > 1$, $Q_+(\zeta) \sim O(\zeta^2)$ when $|\zeta| \to \infty$, $\Im m\, \zeta > 0$. Likewise it may be shown that $Q_-(\zeta) \sim O(\zeta^2)$ when $|\zeta| \to \infty$, $\Im m\, \zeta < \kappa_{1,n}$. It is seen that $Q(\zeta) \sim O(1)$ in the strip. The argument is now identical with that of Appendix 2. It may therefore be concluded that the integral function $E(\alpha)$ is zero.

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